Published by IOP Publishing for SISSA

Received: *November 13, 2008*
Accepted: *December 17, 2008*
Published: *December 30, 2008*

A new three-algebra representation for the Superconformal Chern-Simons Theory

M.M. Sheikh-Jabbari

School of Physics, Institute for research in fundamental sciences (IPM),
P.O.Box 19395-5531, Tehran, Iran
E-mail: jabbari @theory.ipm.ac.ir

Abstract: Based on the realization of three-algebras in terms of algebra of matrices and four-brackets [arXiv:0807.1570] we present the notion of $u(N)$-based extended three-algebras, which for $N = 2$ reproduces the Bagger-Lambert three-algebra. Using these extended three-algebras we construct an $su(N) \times su(N)$ Chern-Simons action with explicit SO(8) invariance. The dynamical fields of this theory are eight *complex valued* bosonic and fermionic fields in the bi-fundamental representation of the $su(N) \times su(N)$. For generic N the fermionic transformations, however, close only on a subclass of the states of this theory onto the $3d$, $\mathcal{N} = 6$ superalgebra. In this sector we deal with four complex valued scalars and fermions, our theory is closely related to the ABJM model [arXiv:0806.1218], and hence it can be viewed as the (low energy effective) theory of N M2-branes. We discuss that our three-algebra structure suggests a picture of open M2-brane stretched between any two pairs of M2-branes. We also analyze the BPS configurations of our model.

Keywords: AdS-CFT Correspondence, M-Theory.

© SISSA 2008

JHEP12(2008)111

Contents

JHEP12(2008)111

1. Introduction

Motivated by the proposal made by J. Schwarz [1], recently Bagger and Lambert [2, 3] and Gustavson [4, 5] have proposed an action for maximally supersymmetric three-dimensional conformal field theory (see [6] for a recent review). This action is basically a supersymmetric Chern-Simons theory, in which instead of the usual Lie-algebraic structures and commutators one deals with a new type of algebra which has a bracket involving three elements of the algebra (rather than two for the commutator). This kind of algebra was hence called *three-algebra*.

The metric three-algebras are defined through a three-bracket structure and a "trace" over the algebra (and hence a metric) and a generalization of the Jacobi Identity, the *fundamental identity*. According to the *three-algebra no-go theorem* [7] the only three-algebra which has a positive definite norm is either $so(4)$ or direct sums of a number of $so(4)$'s. In this sense the original Bagger-Lambert-Gustavson (BLG) theory is rather unique [6].

The restriction were bypassed relaxing the positive norm condition and it was shown [8 − 10] (see also [11]) that allowing a single negative eigenvalue in the metric one has the possibility of constructing three-algebras based on any Lie-algebra. The BLG theory based on these *Lorentzian* three-algebras, due to the negative norm in the metric has pathologic ghost-type fields (fields with negative kinetic energy). Despite of the proposals and arguments that these ghost-type fields are not harmful to the unitarity of the theory [12 − 16] the connection of these theories to that of multi M2-brane is not clear yet.

The $3d$, $\mathcal{N} = 8$ Super-Conformal Field Theory (SCFT) is expected to arise from the low energy limit of a system of multi M2-branes and be dual to M-theory on $AdS_4 \times S^7$ [17]. With this motivation and the difficulties with extending the BLG theory and their usual three-algebras, inspired by ideas in [18],[1] Aharony, Bergman, Jafferis and Maldacena, (ABJM) [19] constructed an $\mathcal{N} = 6$ $u(N) \times u(N)$ supersymmetric Chern-Simons theory at level k with matter fields in the bi-fundamental of the gauge group. This theory is proposed to be describing N M2-branes on a Z_k orbifold or M-theory on $AdS_4 \times S^7/Z_k$. In [20] it was shown that the ABJM theory has a representation in terms of the BLG theory with a "generalized" notion of three-algebra.

In this paper we attempt in writing an explicit action for the $3d$, $\mathcal{N} = 8$ $su(N) \times su(N)$ Chern-Simons theory. To this end we start from the BLG theory but with a new *extended three-algebra*. Using the four-bracket representation for the three-algebras introduced in [15] (see also [21]) we give a matrix representation for the extended three-algebra in terms of $2N \times 2N$ Hermitian matrices. The underlying $su(2N)$ algebra has an $su(N) \times su(N)$ subalgebra. Utilizing this matrix representation we show that the BLG theory with the above "$u(N)$-based extended three-algebra" is equivalent to a $3d$ $su(N) \times su(N)$ Chern-Simons action. We show that for the $N = 2$ case our extended three-algebra reproduces two copies of the Bagger-Lambert three-algebra. In this action, which for generic N has

[1] In [18] it was shown that the $so(4)$-based BLG theory is nothing but an $su(2) \times su(2)$ Chern-Simons theory with $\mathcal{N} = 8$ supersymmetry.

JHEP12(2008)111

explicit global SO(8) invariance, we are forced to work with eight *complex valued* scalars and fermions in the bi-fundamental representation of the $su(N) \times su(N)$.

The direct generalization of 16 fermionic transformations of the BLG theory, however, do not close onto a generic configuration of the fields in our theory and hence our theory, despite of being SO(8) invariant, is not an $\mathcal{N} = 8$ theory. One may then ask if there is a subclass or a sector of physical configuration over which all or a subset of fermionic transformations indeed form a supersymmetry algebra. As we will show for generic N the largest of such sectors in the Fock space of the theory is the part which is invariant under $\mathrm{SU}(4) \times \mathrm{U}(1) \in \mathrm{SO}(8)$, and with fermionic transformation parameters restricted to be in $\mathbf{6}_0$ of this $\mathrm{SU}(4) \times \mathrm{U}(1)$. In this sector the bosonic scalar degrees of freedom of the theory are four complex valued fields in $\mathbf{4}_{+1}$ of $\mathrm{SU}(4) \times \mathrm{U}(1)$ in bi-fundamental of $su(N) \times su(N)$ and their complex conjugates, half of our original theory. In this sector the theory exhibits $\mathcal{N} = 6$ supersymmetry which is the largest possible supersymmetry within the class of our models and is hence closely related to the ABJM model [19]. We show that for the special case of $N = 2$, because of the special properties of the $su(2)$ algebra, besides the projection onto the $\mathrm{SU}(4) \times \mathrm{U}(1)$ sector, one has the option of closing all 16 supersymmetry variations by projecting into another invariant sector while keeping the SO(8). In this sense the Bagger-Lambert theory is different than the ABJM theory for $N = 2$.

We propose that our $su(N) \times su(N)$ Chern-Simons theory once projected onto the $\mathrm{SU}(4) \times \mathrm{U}(1)$ sector, describes the low energy theory for N M2-branes on the flat space background. Our construction in terms of $N \times N$ complex valued fields finds a natural suggestive "geometric" picture through *two* pairs of open membranes stretched between any two M2-branes. These two pairs are related by the $3d$ worldvolume parity which is connected with the "projection" onto the $\mathrm{SU}(4) \times \mathrm{U}(1)$ invariant sector in the Fock space described above. This picture sheds light on both the underlying $2N \times 2N$ matrices and $su(2N)$ structure, its $su(N) \times su(N)$ subalgebra and why the projection is necessary to avoid over counting of degrees of freedom.

The paper is organized as follows. In section 2, we review basics of three-algebras and their representation in terms of ordinary matrices and the four-brackets. In section 3, we present the notion of "extended" three-algebra and also the *$u(N)$-based extended three-algebra*, the three-algebra that we propose for N M2-bane theory. In section 4, we construct the BLG theory based on the extended three-algebra and discuss its supersymmetry, gauge symmetry and other global symmetries as well as the behavior under the $3d$ parity. In section 5, we show that our theory is equivalent to an $su(N) \times su(N)$ Chern-Simons gauge theory with explicit SO(8) invariance, while not $\mathcal{N} = 8$ invariant. We discuss its relation to the ABJM model once we restrict our theory to the sector of the Fock space over which the supersymmetry closes to $\mathcal{N} = 6$ algebra. In section 6, the relevance of our model to M2-branes is discussed and the BPS configurations of our model is analyzed. We also show that although the theory for a generic configuration is an $\mathcal{N} = 6$ theory, there are BPS configurations for which the theory can exhibit more fermionic symmetries than is expected from the $\mathcal{N} = 6$ theory. The last section is devoted to summary of our results and discussions. In the appendix A, we have gathered our conventions for the $su(N)$ algebras, their representations and some useful identities among $su(N)$ tensors. In appendix B,

JHEP12(2008)111

we present the arguments proving that within our setting the extended three-algebras are only limited to the one generated through $N \times N$ representation of the $u(N)$ algebra, the "$u(N)$-based extended three-algebras". In appendix C, we show that our $u(2)$-based extended three-algebra is a double cover of the $so(4)$-based Bagger-Lambert three-algebra. In appendix D, we show compatibility of the fermionic variations with the $3d$ parity.

2. Preliminaries of three-algebras

In this section we very briefly introduce the notion of three-algebras and some basic facts about them. We then discuss a representation of three-brackets of the three-algebras in terms of four-brackets and ordinary associative algebra of matrices.

2.1 Introduction to three-algebras

The three-algebra $\mathcal{A}_3$ is an algebraic structure defined through the three-bracket $[\![\ ,\ ,\ ]\!]$[2]

$$[\![\Phi_1, \Phi_2, \Phi_3]\!] \in \mathcal{A}_3, \quad \text{for any } \Phi_i \in \mathcal{A}_3, \tag{2.1}$$

where

$$[\![\Phi_1, \Phi_2, \Phi_3]\!] = -[\![\Phi_2, \Phi_1, \Phi_3]\!] = -[\![\Phi_1, \Phi_3, \Phi_2]\!] = -[\![\Phi_3, \Phi_2, \Phi_1]\!] \tag{2.2}$$

The three-bracket should satisfy an analog of the Jacobi identity, the *fundamental identity* [22]:

$$\begin{aligned}\mathcal{K}_{ij;klm} &\equiv [\![\Phi_i, \Phi_j, [\![\Phi_k, \Phi_l, \Phi_m]\!]]\!] \\ &= [\![[\![\Phi_i, \Phi_j, \Phi_k]\!], \Phi_l, \Phi_m]\!] + [\![[\![\Phi_i, \Phi_j, \Phi_l]\!], \Phi_m, \Phi_k]\!] + [\![[\![\Phi_i, \Phi_j, \Phi_m]\!], \Phi_k, \Phi_l]\!].\end{aligned} \tag{2.3}$$

As we can see $\mathcal{K}_{ij;klm}$ is anti-symmetric under exchange of first two as well as the last three indices. We equip this algebra with a product $\bullet$ and a Trace

$$Tr(\Phi_1 \bullet \Phi_2) = Tr(\Phi_2 \bullet \Phi_1) \in \mathbb{C} \tag{2.4}$$

with a "by-part integration" property

$$Tr(\Phi_1 \bullet [\![\Phi_2, \Phi_3, \Phi_4]\!]) = -Tr([\![\Phi_1, \Phi_2, \Phi_3]\!] \bullet \Phi_4). \tag{2.5}$$

Φ_i's are generically complex valued and we can define the Hermitian conjugation over the algebra and its three-bracket:

$$[\![\Phi_1, \Phi_2, \Phi_3]\!]^\dagger = [\![\Phi_1^\dagger, \Phi_2^\dagger, \Phi_3^\dagger]\!]\ . \tag{2.6}$$

Let T^α denote a complete basis in $\mathcal{A}_3$, i.e. $\forall \Phi \in \mathcal{A}_3,\ \ \Phi = \Phi_\alpha T^\alpha$, then (2.1) implies that

$$[\![T^\alpha, T^\beta, T^\gamma]\!] = f^{\alpha\beta\gamma}{}_\rho T^\rho \tag{2.7}$$

[2] Since we will be working with usual matrices and will be using the usual commutators of matrices and also introduce the new notion of four-brackets, we will use $[\![\ ,\ ,\ ]\!]$ for three-algebra brackets and usual brackets for matrix valued objects, either commutator or four-brackets.

JHEP12(2008)111

and

$$Tr(T^\alpha \bullet T^\beta) \equiv h^{\alpha\beta} \tag{2.8}$$

defines the metric $h^{\alpha\beta}$ on $\mathcal{A}_3$. The metric $h^{\alpha\beta}$ can in general have positive or negative eigenvalues, however, $h^{\alpha\beta}$ is always taken to be non-degenerate and invertible. Noting (2.2) and (2.5),

$$f^{\alpha\beta\gamma\delta} \equiv f^{\alpha\beta\gamma}{}_\lambda h^{\lambda\delta},$$

is totally anti-symmetric four-index structure constant. The fundamental identity in terms of the structure constant f is written as

$$f^{\alpha\beta\gamma}{}_\lambda f^{\delta\eta\lambda}{}_\mu + f^{\alpha\beta\delta}{}_\lambda f^{\eta\gamma\lambda}{}_\mu + f^{\alpha\beta\eta}{}_\lambda f^{\gamma\delta\lambda}{}_\mu = f^{\gamma\delta\eta}{}_\lambda f^{\alpha\beta\lambda}{}_\mu. \tag{2.9}$$

It has been shown that [7] for Euclidean case, when $h^{\alpha\beta}$ is positive definite, (2.9) has only a single solution $f^{\alpha\beta\gamma\delta} \propto \epsilon^{\alpha\beta\gamma\delta}$, while when $h^{\alpha\beta}$ is Lorentzian (when h has a single negative eigenvalue), one can associate a three-algebra structure to any Lie-algebra [8 – 10]. In this case the fundamental identity reduces to the Jacobi identity of the algebra and the structure constant of the three-algebra is expressed in terms of the structure constant of the underlying Lie-algebra.

We would like to comment that for the Euclidean and the Lorentzian cases one can choose a Hermitian basis T^α for which the structure constants $f^{\alpha\beta\gamma\delta}$ are real valued.

2.2 Four-bracket representation for three-algebras

As discussed in [15] one may give a representation of three-algebras in terms of ordinary algebra of matrices. To that end we need to give a four-bracket realization for the three-brackets of the three-algebra:

$$[\![A_1, A_2, A_3]\!] \equiv [\hat{A}_1, \hat{A}_2, \hat{A}_3, T] \tag{2.10}$$

where the hatted quantities are just normal matrices and T is a matrix which anticommutes with all the other elements of the algebra

$$\{A_i, T\} = 0\ . \tag{2.11}$$

The four-bracket is defined as antisymmetrized product of the elements appearing inside, that is

$$\begin{aligned}[\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4] &= \frac{1}{4!}\epsilon^{ijkl}\hat{A}_i\hat{A}_j\hat{A}_k\hat{A}_l \\ &= \frac{1}{4!}\Big(\{[\hat{A}_1, \hat{A}_2], [\hat{A}_3, \hat{A}_4]\} - \{[\hat{A}_1, \hat{A}_3], [\hat{A}_2, \hat{A}_4]\} + \{[\hat{A}_1, \hat{A}_4], [\hat{A}_2, \hat{A}_3]\}\Big)\ .\end{aligned} \tag{2.12}$$

The fundamental identity (2.3) in terms of the four-bracket takes the form[3]

$$\begin{aligned}&[[A_1, A_2, B_1, T], B_2, B_3, T] + [B_1, [A_1, A_2, B_2, T], B_3, T] \\ &\qquad + [B_1, B_2, [A_1, A_2, B_3, T], T] = [A_1, A_2, [B_1, B_2, B_3, T], T],\end{aligned} \tag{2.13}$$

[3]Hereafter we will drop the hats on any matrix A.

JHEP12(2008)111

for any element A_i and B_i in the algebra. Working with matrices, we can choose the trace over the matrices as the natural trace over our three-algebra.

It is evident that with the above definitions not all arbitrary sets of matrices satisfy the closure (2.1) and fundamental identity (2.13). It is, however, immediate to check that within our matrix representation and the four-bracket, the trace condition (2.5) and the Hermitian conjugation (2.6) (if $T = T^\dagger$) are automatically satisfied. In [15] it was shown that the only set of matrices which satisfy the closure and fundamental identity requirements as stated above, are the "$so(4)$-based" algebras (where A_i's and T are respectively taken to be $N \times N$ representation of $so(4)$ Dirac γ-matrices and the γ^5), compatible with the three-algebra no-go theorem [7].

3. $u(N)$-based extended three-algebras

As was argued by Bagger and Lambert [3] the requirement of fundamental identity for the three-algebras is demanded by the "gauge symmetry" as well as the closure of the supersymmetry algebra in the BLG theory. The "Tr" operation (and hence the metric), however, is needed to construct "gauge invariant" physical observables. Given the restrictions on the construction of the three-algebras one is hence motivated to see if the notion of fundamental identity and/or the closure condition can be relaxed or extended in such a way that the gauge invariance and the $\mathcal{N} = 8$ supersymmetry algebra requirements are met, while allowing for further possibilities of three-algebras.

In [15] one such possibility, which were dubbed as the *relaxed three-algebras*, was explored. There, it was noted that by the addition of a "spurious" part of the algebra of matrices one can relax the closure condition and the fundamental identity holds up to the "spurious" parts, while keeping the virtues resulting from those properties. In this way an explicit matrix representation for the Lorentzian three-algebras were given and was shown that the Lorentzian three-algebra is a unique outcome of the non-empty spurious part of the algebra [15].

Here we study yet another way of extending the notion of the three-algebras by revisiting the notion of the fundamental identity. As it will become clear in the next sections, what is needed to ensure the gauge symmetry closure is not the strict form of the fundamental identity given in (2.3) or (2.13). A similar observation has also been made in [20]. In [20], however, the focus was working with non-totally antisymmetric three-brackets, whereas in our case the brackets are still totally antisymmetric and the implementation of the fundamental identity is modified. This will become clear in this section.

In what follows based on the appropriate notion of extended fundamental identity, we construct the *extended three-algebra*, using our four-bracket and matrix representation introduced in the previous subsection.

3.1 Construction of the extended three-algebras

To start we assume that the complete basis for the three-algebra is of the following form

$$T^M \in \{T_+^A,\ T_-^A,\ T\}\,, \tag{3.1}$$

JHEP12(2008)111

with

$$T^A_\pm = t^A \otimes \sigma^\pm, \qquad T = \mathbb{1}_N \otimes \sigma^3\,, \tag{3.2}$$

where t^A are (yet to be specified) set of $N \times N$ *Hermitian* matrices and $\sigma^\pm$, σ^3 are the 2×2 Pauli matrices

$$[\sigma^+, \sigma^-] = \sigma^3, \quad [\sigma^3, \sigma^\pm] = \pm 2\sigma^\pm, \quad \{\sigma^+, \sigma^-\} = \mathbb{1}_{2\times 2}. \tag{3.3}$$

Since t^A's are Hermitian,

$$(T^A_+)^\dagger = T^A_- \,. \tag{3.4}$$

With the above it is clear that

$$\{T^A_\pm, T\} = 0\,, \qquad T^2 = \mathbb{1}_{2N\times 2N}, \quad [T, T^A_\pm] = \pm 2T^A_\pm\,, \tag{3.5}$$

moreover,

$$T^A_+ T^B_+ = T^A_- T^B_- = 0\,. \tag{3.6}$$

We normalize our basis such that

$$Tr(T^A_+ T^B_-) = Tr(T^A_- T^B_+) = \frac{1}{2}\delta^{AB}\,. \tag{3.7}$$

Let us consider the most general four-bracket $[T^M, T^N, T^P, T]$. It is evident that if any of T^M, T^N or T^P is T the bracket vanishes. We hence remain with four types of four-brackets, two of them are those which only involve T^A_+ or T^A_- identically vanish,

$$[T^A_+, T^B_+, T^C_+, T] = [T^A_-, T^B_-, T^C_-, T] = 0, \tag{3.8}$$

where we have used $(\sigma^+)^2 = (\sigma^-)^2 = 0$ and the definition of the four-bracket. The other two are those with two T^A_+ and one T^A_- or two T^A_- and one T^A_+, which are related by Hermitian conjugation

$$\left([T^A_+, T^B_-, T^C_+, T]\right)^\dagger = [T^A_-, T^B_+, T^C_-, T]\,, \tag{3.9}$$

where we have used (3.4). Therefore there is only a single type of independent four-bracket.

Using straightforward algebra of Pauli matrices and the definition of the four-bracket we have

$$[T^A_+, T^B_-, T^C_+, T] = \frac{-1}{6}\left(t^A t^B t^C - t^C t^B t^A\right) \otimes \sigma^+\,. \tag{3.10}$$

3.2 Closure condition

Demanding the closure of the four-bracket over the set of T^A_+ and T^A_- requires that

$$\frac{-1}{6}\left(t^A t^B t^C - t^C t^B t^A\right) = f^{ABC}{}_D\, t^D \tag{3.11}$$

for some numeric coefficients $f^{ABC}{}_D$. If we choose to work with t^A which are generators of a (semi-simple) Lie-algebra,[4] the above closure condition (3.11) is very restrictive and

[4] It is worth noting that this is a working assumption and not a necessary one.

JHEP12(2008)111

uniquely fixes this algebra to be a $u(N)$ (for arbitrary N). Moreover, it also requires t^A's to be in the $N \times N$ fundamental representation of the $u(N)$ algebra. In other words, the closure condition (3.11) is only satisfied for the algebras which are their own enveloping algebra and $u(N)$ in the $N \times N$ representation is the only such algebra. In the appendix B, we present a proof of this statement. These algebras will hence be called $u(N)$-based (extended) three-algebras. Using (3.11) we have

$$\begin{aligned} &[T_+^A, T_-^B, T_+^C, T] = f^{ABC}{}_D T_+^D \ , \\ &[T_-^A, T_+^B, T_-^C, T] = -f^{ABC}{}_D T_-^D \ . \end{aligned} \tag{3.12}$$

In the second identity we have used the fact that, noting (3.11) and hermiticity of t^A's, f is pure imaginary.

Using (2.5) we have

$$f^{ABCD} = -2\ Tr\left([T_+^A, T_-^B, T_+^C, T_-^D]T\right) \ . \tag{3.13}$$

The above explicitly shows that

$$f^{ABCD} = -f^{CBAD} = -f^{ADCB} = +f^{CDAB} = -f^{BADC} = -(f^{ABCD})^* \ . \tag{3.14}$$

For the last two identities we have used the fact that f is pure imaginary. From (3.10) and that $t^A t^A \propto \mathbb{1}$, it is readily seen that

$$\sum_A f^{AABC} = 0 \ .$$

We would like to comment that f^{ABCD} with the above symmetry properties may be viewed as the structure constant of a new type (or "generalized") three-algebra [20, 23, 24]. The three-bracket of these generalized three-algebras are hence not totally antisymmetric and as a consequence their fundamental identity is expressed in a bit different way than (2.3). Our notion and realization of the extended three-algebras, although looking similar to the constructions discussed [20, 23, 24], has its own specific features. In particular, as is explicitly seen from the definition of our brackets (2.10) and (2.12), our four-brackets are antisymmetric under exchange of any two elements. Therefore, in the M, N, P basis and before expansion in $T_\pm^A$, T basis, the structure constant $\hat{f}$,

$$\hat{f}^{MNPQ} \equiv -Tr([T^M, T^N, T^P, T^Q]T),$$

is totally antisymmetric. Moreover, we have an explicit matrix representation and $u(N)$ algebra has a distinguished role in our setting.

For the specific choice of $u(N)$ basis given in the appendix A (where t^a's are generators of $su(N)$ part of $u(N)$ and $t^0 \propto \mathbb{1}$ is its $u(1)$ part) one can show that:

$$f^{00ab} = 0, \tag{3.15a}$$

$$f^{0abc} = f^{a0bc} = f^{ab0c} = f^{abc0} = \frac{-i}{6} \cdot \frac{1}{\sqrt{2N}}\ f^{abc}, \tag{3.15b}$$

$$f^{abcd} = \frac{-i}{12}\left(f^{abe}d^{cde} + f^{cde}d^{abe}\right). \tag{3.15c}$$

JHEP12(2008)111

It is worth noting that for the specific case of $N = 2$, the $u(2)$ algebra, $d^{abc} = 0$ and hence $f^{abcd} = 0$. In this case the only non-vanishing components of f are $f^{0abc} \propto \epsilon^{abc}$, $a, b, c = 1, 2, 3$. As it has been shown in appendix B, for the $N = 2$ case one can choose a sector (by working with half of the eight $T^A_\pm$ generators) in which the structure constants become totally antisymmetric. Among the $u(N)$ based (extended) three-algebras the $u(2)$ case is the only one with the possibility of totally antisymmetric structure constant.

3.3 Extended fundamental identity

As discussed (e.g. see [3]) the fundamental identity (2.3) or in its four-bracket presentation (2.13) is necessitated by the gauge invariance and the superalgebra closure of the BLG theory. However, as will become clear in the next section, these conditions might be met through a bit weaker condition than (2.13): It is enough to check the fundamental identity (2.13) for the case when either of A_1, A_2 are of the form of T^A_+ and T^A_- (and not both of them of the form of T^A_+ or T^A_-) while B_i's can be arbitrary. In terms of our basis that is,

$$\begin{aligned} &[[T^A_+, T^B_-, T^M, T], T^N, T^P, T] + [T^M, [T^A_+, T^B_-, T^N, T], T^P, T] \\ &\qquad + [T^M, T^N, [T^A_+, T^B_-, T^P, T], T] = [T^A_+, T^B_-, [T^M, T^N, T^P, T], T], \end{aligned} \tag{3.16}$$

where T^M, T^N, T^P are either T^A_+, T^A_- or T.

Recalling the discussions of sections 3.1 and 3.2, the extended fundamental identity (3.16) for $(T^M, T^N, T^P) = (T^C_+, T^D_+, T^E_+)$ or (T^C_-, T^D_-, T^E_-) is trivially satisfied while it should be checked for $(T^M, T^N, T^P) = (T^C_+, T^D_+, T^E_-)$ or $(T^M, T^N, T^P) = (T^C_+, T^D_-, T^E_-)$ (or in general two plus and a minus or two minus and a plus type generators) cases. These two cases, however, are not independent and are related by complex conjugation. Therefore, we will only need to verify one of these cases which we choose it to be $(T^M, T^N, T^P) = (T^C_+, T^D_-, T^E_+)$. It is straightforward to verify that fundamental identity (3.16) is satisfied for this case. This may be done directly using (3.10) and the associativity of the product of t^A's (without using the fact that t^A's are generators of $u(N)$). Since, as discussed in section 3.2, the closure condition requires that in our extended three-algebras t^A's must be generators of $u(N)$, we call them *$u(N)$-based extended three-algebras*.

It is useful to represent the fundamental identity in terms of the "structure constants" f^{ABCD}:

$$f^{ABGH}\, f^{CDFG} + f^{ABGD}\, f^{CGFH} + f^{ABCG}\, f^{FDGH} = f^{ABFG}\, f^{CDGH}\,. \tag{3.17}$$

Note that the indices on f are lowered and raised by the metric defined in (3.7), i.e. δ_{AB} when we work with A and B indices instead of M and N indices. One can also verify that the above identity is fulfilled using the explicit expression for f given in (3.15) and using the identities given in the appendix A. In the appendix B we show the connection between the Bagger-Lambert three-algebra and the $u(2)$-based extended three-algebra.

4. The SO(8) invariant SCFT action

Since the on-shell matter content of the $3d$, $\mathcal{N} = 8$ SCFT should involve eight *real valued* three-dimensional scalars X^I, $I = 1, 2, \cdots, 8$ in the $\mathbf{8_v}$ of the SO(8) R-symmetry group,

JHEP12(2008)111

eight two component Majorana (real valued) three-dimensional fermions Ψ (i.e. they satisfy $\gamma^{012}\Psi = \Psi$) in the $\mathbf{8_s}$ of SO(8), we start with this explicitly SO(8) notation. Unless there can be confusion, here we will suppress both the $3d$ and the R-symmetry fermionic indices. Each of the above physical fields, which will generically be denoted by Φ, are also assumed to be elements of the $u(N)$-based extended three-algebra and hence

$$\Phi = \Phi_M T^M = \Phi^+_A\ T^A_+ + \Phi^-_A\ T^A_- + \Phi_T\ T. \tag{4.1}$$

As argued by Bagger and Lambert [3] and Gustavson [4] to close the $\mathcal{N} > 4$ supersymmetry algebra, besides the above propagating physical fields we need to introduce a non-propagating gauge field with a Chern-Simons action. The gauge field should have two three-algebra indices, i.e.

$$A_\mu = \frac{1}{2} A_{\mu AB}\ [T^A_+, T^B_-]\ . \tag{4.2}$$

We would like to emphasize that the $A_{\mu AB}$ components are not anti-symmetric under the exchange of A and B indices.

As we will show in this section, the three-algebra with the extended notion of the fundamental identity (3.16) is enough to ensure the closure of the gauge transformations. The extended fundamental identity, however, is not enough to guarantee the closure of the SO(8) *covariant* (i.e. $\mathcal{N} = 8$) supersymmetry transformations. As a result we are forced to close the supersymmetry onto a smaller set of states. As we will show the largest set of such states keep SU(4) $\simeq$ SO(6) $\in$ SO(8) (i.e. $\mathcal{N} = 6$) supersymmetry.

4.1 The BLG Lagrangian in terms of four-brackets

As discussed in [15] one can represent the BLG theory in terms of the four-brackets. This representation explicitly exhibits the SO(8) invariance of the theory. Here we take the physical fields and the four-brackets to be in the $u(N)$-based extended three-algebra discussed in the previous section.

The gauge invariant action with explicit SO(8) symmetry.

$$\begin{aligned} S = \int d^3x\ Tr\Big[& -\frac{1}{2} D_\mu X^I D^\mu X^I - \frac{1}{2.3!}[X^I, X^J, X^K, T][X^I, X^J, X^K, T] \\ & + \frac{i}{2}\bar{\Psi}\gamma^\mu D_\mu \Psi - \frac{i}{4}[\bar{\Psi}, X^I, X^J, T]\Gamma^{IJ}\Psi \\ & + \frac{1}{2}\epsilon^{\mu\nu\rho}\left(A_{\mu AB}\partial_\nu A_{\rho CD} T^D_- + \frac{2}{3} A_{\mu AB} A_{\nu CD} A_{\rho EF}[T^D_-, T^E_+, T^F_-, T]\right)[T^A_+, T^B_-, T^C_+, T]\Big], \end{aligned} \tag{4.3}$$

where the trace is over $2N \times 2N$ matrices and

$$D_\mu \Phi \equiv \partial_\mu \Phi - A_{\mu AB}[T^A_+, T^B_-, \Phi, T]\ . \tag{4.4}$$

In terms of the components it is

$$(D_\mu\Phi)_T = \partial_\mu \Phi_T \tag{4.5a}$$

$$(D_\mu\Phi)^+_D = \partial_\mu \Phi^+_D - f^{ABC}{}_D A_{\mu AB}\ \Phi^+_C \tag{4.5b}$$

$$(D_\mu\Phi)^-_D = \partial_\mu \Phi^-_D + f^{ABC}{}_D A_{\mu BA}\ \Phi^-_C\,, \tag{4.5c}$$

JHEP12(2008)111

where in (4.5c) we have used the properties of f^{ABCD} (3.14).

With the above definition it is seen that if $\Phi = \Phi^\dagger$, then $D_\mu\Phi = (D_\mu\Phi)^\dagger$. Moreover,

$$A^*_{\mu AB} = -A_{\mu BA}\,, \tag{4.6}$$

where $*$ is the complex conjugation. In terms of the gauge field A_μ (4.2), i.e. $A^\dagger_\mu = -A_\mu$. As in [3] it is useful to define a new gauge field

$$\tilde{A}_{\mu CD} = f^{ABCD} A_{\mu AB}\ . \tag{4.7}$$

In terms of $\tilde{A}_\mu$ the covariant derivatives take the form

$$(D_\mu\Phi)^+_A = \partial_\mu\Phi^+_A - \tilde{A}_{\mu BA}\ \Phi^+_B, \qquad (D_\mu\Phi)^-_A = \partial_\mu\Phi^-_A + \tilde{A}_{\mu AB}\ \Phi^-_B. \tag{4.8}$$

It is worth noting that the $\tilde{A}_\mu$ gauge field, similarly to $A_{\mu AB}$, has only $[T^A_+, T^B_-]$ components.

Gauge transformations.

$$\delta_{\text{gauge}}\Phi_T = 0 \tag{4.9a}$$

$$\delta_{\text{gauge}}\Phi^+_A = \tilde{\Lambda}_{BA}\Phi^+_B\,, \qquad \delta_{\text{gauge}}\Phi^-_A = -\tilde{\Lambda}_{AB}\Phi^-_B\,, \tag{4.9b}$$

$$\delta_{\text{gauge}}\tilde{A}_{\mu AB} = \partial_\mu\tilde{\Lambda}_{AB} + \left(\tilde{A}_{\mu AC}\tilde{\Lambda}_{CB} - \tilde{\Lambda}_{AC}\tilde{A}_{\mu CB}\right)\ . \tag{4.9c}$$

Note that like the $\tilde{A}_\mu$, $\tilde{\Lambda}$ has only components along $[T^A_+, T^B_-]$.

From the above it is readily seen that

$$\delta_{\text{gauge}}(D_\mu\Phi)^+_A = \tilde{\Lambda}_{BA}(D_\mu\Phi)^+_B\,, \qquad \delta_{\text{gauge}}(D_\mu\Phi)^-_A = -\tilde{\Lambda}_{AB}(D_\mu\Phi)^-_B\ . \tag{4.10}$$

The action (4.3) is invariant under the above gauge transformations provided that

$$\delta_{\text{gauge}}\left([\Phi_1, \Phi_2, \Phi_3, T]\right) = [\delta_{\text{gauge}}\Phi_1, \Phi_2, \Phi_3, T] + [\Phi_1, \delta_{\text{gauge}}\Phi_2, \Phi_3, T] + [\Phi_1, \Phi_2, \delta_{\text{gauge}}\Phi_3, T]. \tag{4.11}$$

This identity holds as a result of the extended fundamental identity (3.16), once we recall that the gauge transformations parameter Λ has one plus type and one minus type T^A generators. As a result of the extended fundamental identity one can also show that

$$D_\mu\left([\Phi_1, \Phi_2, \Phi_3, T]\right) = [D_\mu\Phi_1, \Phi_2, \Phi_3, T] + [\Phi_1, D_\mu\Phi_2, \Phi_3, T] + [\Phi_1, \Phi_2, D_\mu\Phi_3, T]. \tag{4.12}$$

Eqs.(4.11) and (4.12) are nothing but the statement of closure of the gauge symmetry algebra of the action (4.3).

So far we have presented a theory which enjoys the gauge symmetry (4.9) as well as global SO(8) and $3d$ Poincaré invariance. The propagating bosonic degrees of freedom of this theory are X^I_T, $(X^I)^+_A$, $(X^I)^-_A$. X^I_T are eight real *free* scalars which decouple from the rest of the theory. The X^I_T piece, together with its fermionic counterpart Ψ_T form a trivial $\mathcal{N} = 8$ superconformal theory (with the explicit supersymmetry transformation given in the next subsection). Hereafter, we will hence ignore the Φ_T piece by simply setting them to zero. $(X^I)^+_A = \left((X^I)^-_A\right)^*$ which are elements of $N \times N$ matrices for the $u(N)$-based algebra, parameterize $8N^2$ complex (or $8 \cdot 2N^2$ real) scalars. However, the $\mathcal{N} = 8$ theory is expected to have real valued scalars. As we will see the closure of the supersymmetry and parity invariance of the physical Fock space of the theory should be used to reduce this extra degrees of freedom.

JHEP12(2008)111

4.2 Parity invariance

The $3d$, $\mathcal{N}=8$ theory is expected to be invariant under the $3d$ parity transformations $x^0, x^1 \to x^0, x^1$ and $x^2 \to -x^2$. The parity invariance of the (twisted) Chern-Simons term implies that under parity

$$\tilde{A}_{0AB}, \tilde{A}_{1AB} \longrightarrow -\tilde{A}_{0BA}, -\tilde{A}_{1BA}, \qquad \tilde{A}_{2AB} \longrightarrow +\tilde{A}_{2BA} . \tag{4.13}$$

Recalling (4.6), that is

$$\tilde{A}_{\mu AB} \xleftrightarrow{\text{parity}} (\tilde{A}^p_{\mu AB})^* , \tag{4.14}$$

where by $A^p_{\mu AB}$ we mean a vector with components $A_{0AB},\ A_{1AB},\ -A_{2AB}$.

The parity invariance of the kinetic terms, as well as the interaction terms imply that under parity one should exchange the plus and minus components, for the scalar fields that is,

$$(X^I)^+_A \xleftrightarrow{\text{parity}} (X^I)^-_A , \tag{4.15}$$

and for $3d$ fermions

$$\Psi^+_A \xleftrightarrow{\text{parity}} \gamma^2 \Psi^-_A . \tag{4.16}$$

parity $T^A_+ \longleftrightarrow T^A_-$, $T \to -T$. It is useful to introduce action of the parity on the X^I, Ψ and A_μ fields:

$$\begin{aligned} (X^I)_{\text{parity}} &= (X^I)^-_A\ T^A_+ + (X^I)^+_A\ T^A_- \\ (\Psi)_{\text{parity}} &= \gamma^2 \Psi^-_A\ T^A_+ + \gamma^2 \Psi^+_A\ T^A_- \\ (A_\mu)_{\text{parity}} &= \frac{1}{2} A^p_{\mu AB} [T^A_-, T^B_+] . \end{aligned} \tag{4.17}$$

(Note that, as discussed earlier, we have set the X_T and Ψ_T components to zero.) Using the above and (3.14) one can show that

$$([\Phi_1, \Phi_2, \Phi_3, T])_{\text{parity}} = -[(\Phi_1)_{\text{parity}}, (\Phi_2)_{\text{parity}}, (\Phi_3)_{\text{parity}}, T] , \tag{4.18}$$

where Φ_i are either X^I or Ψ. With these and noting that $\bar{\Psi}\Psi$ is a pseudoscalar [33] one can show that the action (4.3) is invariant under parity. Although the action (4.3) is parity invariant, the physical fields X^I in general are not.

We point out that if under parity the gauge parameter $\tilde{\Lambda}_{AB}$ transforms as $\tilde{\Lambda}_{AB} \to -\tilde{\Lambda}_{BA}$, the gauge transformations (4.9) are compatible with the parity. As discussed, among the gauge field components $\tilde{A}_{\mu AB}$, the antisymmetric part

$$\tilde{A}_{\mu[AB]} = \frac{1}{2}(\tilde{A}_{\mu AB} - \tilde{A}_{\mu BA}) \tag{4.19}$$

transforms as a vector, and the symmetric part

$$\tilde{A}_{\mu\{AB\}} = \frac{1}{2}(\tilde{A}_{\mu AB} + \tilde{A}_{\mu BA}) , \tag{4.20}$$

transforms as a pseudovector.

JHEP12(2008)111

It is worth noting that, as can be seen from (4.1) and (4.2), the action (4.3) is invariant under another global U(1) symmetry, the U(1)$_\lambda$ symmetry: $T^A_\pm \longrightarrow e^{\mp i\lambda}T^A_\mp$, while keeping Φ (4.1) and A_μ (4.2) invariant, explicitly that is,

$$\Phi^\pm_A \longrightarrow e^{\pm i\lambda}\Phi^\pm_A\,, \qquad \Phi_T \to \Phi_T\,, \qquad A_{\mu AB} \to A_{\mu AB}\ . \tag{4.21}$$

The parity changes the sign of the charge under the U(1)$_\lambda$ symmetry. We will comment on U(1)$_\lambda$ further in sections 5 and 6. We also note that $\sigma^\pm$, σ^3 form an $su(2)$ algebra and the U(1)$_\lambda$ and parity are forming an $O(2)$ automorphism of this $su(2)$ algebra.

4.3 Supersymmetry transformations and their closure

After discussing the gauge and parity invariance of our theory, we now discuss its supersymmetry. Since the action (4.3) is essentially the Bagger-Lambert action [3], and recalling that our four-brackets are totally antisymmetric with the trace property (2.5), we propose the following fermionic (or supersymmetry) transformations

$$\delta X^I = i\bar\epsilon\Gamma^I\Psi \tag{4.22a}$$

$$\delta\Psi = D_\mu X^I\Gamma^I\gamma^\mu\epsilon - \frac{1}{6}[X^I, X^J, X^K, T]\Gamma^{IJK}\epsilon \tag{4.22b}$$

$$\delta\tilde A_{\mu AB} = if_{ABCD}\ \bar\epsilon\gamma_\mu\Gamma^I\left((X^I)^+_C\Psi^-_D - (X^I)^-_D\Psi^+_C\right). \tag{4.22c}$$

The fermionic transformation parameter ϵ is a $3d$ anti-Majorana fermion

$$\gamma^{012}\epsilon = -\epsilon\,, \tag{4.23}$$

and is in $\mathbf{8}_c$ of SO(8) (in contrast with Ψ which is in $\mathbf{8}_s$).

As first step we check if the above transformations keep the action (4.3) invariant. The variation of the action under the above transformations is

$$\begin{aligned}\delta S &= \int d^3x\ Tr\left(E.o.M_{X^I}\ \delta X^I + E.o.M_\Psi\ \delta\Psi\right) + E.o.M_{A_{\mu AB}}\ \delta A_{\mu AB} + \partial_\mu J^\mu\,,\\ J^\mu &= Tr\left(-D^\mu X^I\delta X^I + i\bar\Psi\gamma^\mu\delta\Psi + \epsilon^{\mu\nu\alpha}A_\nu\delta\tilde A_\alpha\right)\,,\end{aligned} \tag{4.24}$$

where the first three terms vanish on the solutions of equations of motion and J^μ after some algebraic manipulations takes the form

$$J^\mu = i\bar\epsilon\left(-\gamma^{\mu\nu}\tilde A_{\nu CD}\Gamma^K\left((X^K)^+_C\Psi^-_D - (X^K)^-_D\Psi^+_C\right) - \frac{1}{6}\gamma^\mu Tr([X^I, X^J, X^K, T]\Psi)\Gamma^{IJK}\right). \tag{4.25}$$

For the invariance of the action $\partial_\mu J^\mu$ must vanish for any arbitrary ϵ. This can, however, happen in a specific gauge. It is straightforward to check that if f^{ABCD} were totally antisymmetric then in the gauge $2\gamma^\nu\tilde A_{\nu AD} = 3f_{ABCD}\Gamma^{IJ}(X^I)^-_B(X^J)^+_C$, $\partial_\mu J^\mu$ would vanish when sandwiched between any two ϵ-type (i.e $3d$ anti-Majorana and in $\mathbf{8}_c$ of SO(8)) fermions. For our case, however, f^{ABCD} is not totally anti-symmetric and in the above

JHEP12(2008)111

gauge $\partial_\mu J^\mu$ does not vanish.[5] As will become clear momentarily we choose to work in the gauge where

$$\gamma^\nu \tilde{A}_{\nu AB} = +f_{ACDB}\Gamma^{IJ}(X^I)^-_C(X^J)^+_D, \tag{4.26}$$

when sandwiched between any two ϵ-type fermions. In this gauge we have

$$\delta S = \int \partial_\mu \Big(i\bar{\epsilon}\gamma^\mu \big(\Gamma^I (X^I)^-_A \Psi^-_B \chi^+_{AB} - \Gamma^I (X^I)^+_A \Psi^+_B \chi^-_{AB}\big)\Big) \tag{4.27}$$

where

$$\chi^+_{AB} \equiv f_{ACBD}\, \Gamma^{JK}\, (X^J)^+_C\, (X^K)^+_D\,, \qquad \chi^-_{AB} \equiv f_{ACBD}\, \Gamma^{JK}\, (X^J)^-_C\, (X^K)^-_D\,. \tag{4.28}$$

Invariance of the action then demands that $\chi^\pm = 0$. As we will see closure of the fermionic transformations onto the $3d$ super-Poincaré algebra again demands vanishing of $\chi^\pm$, the condition which will be satisfied for a specific subset of fermionic transformations once the degrees of freedom are also restricted to certain subsector of SO(8) states.

4.3.1 Closure of supersymmetry algebra

As a parallel but equivalent analysis, we also study the closure of two successive fermionic transformations on the fields in our action. The closure of the (on-shell) $\mathcal{N} = 8$ (that is, 16 on-shell supersymmetries) demands that two successive supersymmetry transformations of X^I, Ψ and the gauge field $A_{\mu AB}$, up to gauge transformation and upon using the equations of motion, on the *physical Fock space of the theory* must close onto the $3d$ Poincaré [3].

Our supersymmetry transformations are formally the same as those introduced in [3] and [20], once they are represented in terms of three-brackets, two successive supersymmetry transformations lead to the same results as in [3, 20] and most of the analysis are the same as those appeared in [3, 20]. Therefore we do not present the details of the computations and only stress the points of difference. Three closure conditions should be verified:[6]

- Closing the supersymmetry on the scalars we find [3]

$$[\delta_1, \delta_2]X^I = v^\mu D_\mu X^I - V_{JK}[X^I, X^J, X^K, T]\,, \tag{4.29}$$

  where

$$v^\mu = -2i\bar{\epsilon}_2\gamma^\mu\epsilon_1, \qquad V_{JK} = -i\bar{\epsilon}_2\Gamma_{JK}\epsilon_1\,. \tag{4.30}$$

  Let us now consider the T^A_+ and T^A_- components. We note that the T^A_+ component of $V_{JK}[X^I, X^J, X^K, T]$ involves both $(X^I)^+_B$ and $(X^I)^-_B$ components, while the T^A_+ component of $D_\mu X^I$ is only involving $(X^I)^A_+$ (cf. (4.5)).[7] Explicitly,

$$\begin{aligned}
[\delta_1,\delta_2](X^I)^+_D &= v^\mu\partial_\mu(X^I)^+_D + \big(\tilde{\Lambda}_{AD} - v^\mu\tilde{A}_{\mu AD}\big)(X^I)^+_A - i\bar{\epsilon}_2\chi^+_{AD}\epsilon_1\ (X^I)^-_A \\
[\delta_1,\delta_2](X^I)^-_D &= v^\mu\partial_\mu(X^I)^-_D - \big(\tilde{\Lambda}_{DA} - v^\mu\tilde{A}_{\mu DA}\big)(X^I)^-_A + i\bar{\epsilon}_2\chi^-_{AD}\epsilon_1\ (X^I)^+_A,
\end{aligned} \tag{4.31}$$

[5] The point that with f^{ABCD} which is not totally antisymmetric we cannot keep 16 supersymmetries were mentioned in [20] and further emphasized to us by N. Lambert.

[6] We would like to thank Neil Lambert for his fruitful and critical comments on the closure of supersymmetry in our model.

[7] Although very similar our case, the extra term proportional to $(X^I)^\pm$ in the variation of $(X^I)^\mp$ do not happen in the analysis of [20] because, unlike ours, their bracket is not totally anti-symmetric.

JHEP12(2008)111

where

$$\tilde{\Lambda}_{AD} \equiv 2f_{ABCD}\, V_{JK}\, (X^J)^-_B (X^K)^+_C\,, \tag{4.32}$$

and $\chi^\pm$ are defined in (4.28).

Due to the presence of the χ terms, it is not possible to close $[\delta_1, \delta_2](X^I)^+_A$ onto translations (up to gauge transformations). A similar result is also true for the T^A_- components.

Working in the gauge demanded by the invariance of the action (cf. discussions of the opening of section 4.3),

$$v^\mu \tilde{A}_{\mu AD} = \tilde{\Lambda}_{AD}, \tag{4.33}$$

and we remain with

$$\begin{aligned} [\delta_1, \delta_2](X^I)^+_D &= v^\mu \partial_\mu (X^I)^+_D + \chi^+_{AD}\, (X^I)^-_A \\ [\delta_1, \delta_2](X^I)^-_D &= v^\mu \partial_\mu (X^I)^-_D - \chi^-_{AD}\, (X^I)^+_A. \end{aligned} \tag{4.34}$$

That is, the supersymmetry will close only if $\chi^\pm_{AD}$ are vanishing (on the "physical Fock space of the theory"). Recalling that, with the complex valued $(X^I)^\pm_A$ we have introduced twice as much fields, there is the possibility of closing the supersymmetry on the physical Fock space which only involves a specific half of the degrees of freedom. As we show there is indeed such a possibility.

- Closure of supersymmetry on fermions, after using the equation of motion of fermions, leads to [3]

$$[\delta_1, \delta_2]\Psi = v^\mu D_\mu \Psi - V_{JK}\, [\Psi, X^J, X^K, T]\ . \tag{4.35}$$

  The same analysis presented for X^I's also holds for fermions and in (4.33) gauge the above reduces to (4.34) with $(X^I)^\pm_A$ replaced with $\Psi^\pm_A$. Therefore, closure of supersymmetry for fermions demands a similar condition as the scalars, the point to be discussed momentarily.

- The closure of supersymmetry for the gauge fields is more involved. Performing the analysis, we find that in $[\delta_1, \delta_2]\tilde{A}_{\mu AB}$ there is a term proportional to (see eq. (35) of [3])

$$-\frac{i}{3}\left(\bar{\epsilon}_2 \gamma_\mu \Gamma^{IJKL} \epsilon_1\right)\ Tr(X^I[[X^J, X^K, X^L, T], T^A_+, T^B_-, T])\ . \tag{4.36}$$

  This term vanishes for any two arbitrary $3d$ fermions ϵ_1, ϵ_2 and any choice of A, B indices, once we recall the extended fundamental identity (3.16), (2.5) and the totally antisymmetry of Γ^{IJKL}. Following the computations of [3] and using the equation of motion of the gauge field we obtain

$$[\delta_1, \delta_2]\tilde{A}_{\mu AB} = v^\nu \tilde{F}_{\mu\nu AB} - D_\mu \tilde{\Lambda}_{AB}, \tag{4.37}$$

  where

$$\tilde{F}_{\mu\nu AB} = \partial_\mu \tilde{A}_{\nu AB} - \partial_\nu \tilde{A}_{\mu AB} + \tilde{A}_{\mu AC}\tilde{A}_{\nu CB} - \tilde{A}_{\nu AC}\tilde{A}_{\mu CB}.$$

  In the gauge (4.33), we see that $[\delta_1, \delta_2]\tilde{A}_\mu$, closes on translations without any extra $\chi^\pm$-type terms.

JHEP12(2008)111

4.3.2 Projection onto the supersymmetric Hilbert space

Although the supersymmetry transformations are compatible with parity (see appendix C), X^I are not parity invariant and hence the Fock space constructed from operators built upon X^I is not parity invariant. One may hope that the above supersymmetry non-closure will be resolved on the "parity invariant" sector of the Fock space. As can be seen from the closure analysis of previous subsections the supersymmetry closure implies $\chi^{\pm}_{AB} = 0$, which obviously cannot be realized while keeping the SO(8) invariance of the Fock space. We are hence forced to compromise the SO(8) covariance of the states.[8]

The $\chi^{\pm}_{AB} = 0$ condition can, however, be met on a smaller set of states and fermionic (supersymmetry) transformations. It turns out that the largest sector in the Hilbert space of the theory for which $\chi^{\pm}_{AB}$ vanishes is the part which is invariant under $SO(6) \times U(1) \simeq SU(4) \times U(1) \in SO(8)$. To see this we should perform a specific "projection" onto this $SU(4) \times U(1)$ invariant sector. Let us start with the $(X^I)^{\pm}$. Instead of a generic function (operator made) of eight complex valued $(X^I)^{\pm}$ we project onto the functions (states) made out of four complex scalars

$$Z^\alpha = X^\alpha_+ + iX^{\alpha+4}_+, \qquad \bar{Z}_\alpha = (Z^\alpha)^* = X^\alpha_- - iX^{\alpha+4}_-, \qquad \alpha = 1,2,3,4\ . \tag{4.38}$$

It is evident that Z^α and $\bar{Z}_\alpha$ transform as $\mathbf{4}$ and $\bar{\mathbf{4}}$ of SU(4) and under the U(1) $Z_\alpha \to e^{i\xi} Z_\alpha$. To distinguish this U(1) symmetry from the one introduced in (4.21) we denote it by $U(1)_\xi$. That is, e.g. Z_α is in $\mathbf{4}_{+1}$ and $\bar{Z}_\alpha$ in $\bar{\mathbf{4}}_{-1}$ of $SU(4) \times U(1)_\xi$.

As discussed in the end of section 4.2 our $(X^I)^{\pm}$ fields are also charged under the global $U(1)_\lambda$. It is evident that Z_α carry charge $+1$ and $\bar{Z}_\alpha$ charge -1 of the $U(1)_\lambda$; that is, Z_α is in $(+1,+1)$ and $\bar{Z}_\alpha$ in $(-1,-1)$ representation of $U(1)_\lambda \times U(1)_\xi$. We comment that

$$(Z^\alpha)_{\text{parity}} = X^\alpha_- + iX^{\alpha+4}_- \neq \bar{Z}_\alpha$$

and as such under parity the SU(4) and $U(1)_\xi$ representation remains intact while the $U(1)_\lambda$ charge changes sign. $(Z^\alpha)_{\text{parity}}$ and $(\bar{Z}_\alpha)_{\text{parity}}$ are hence respectively in $(-1,+1)$ and $(+1,-1)$ representation of $U(1)_\lambda \times U(1)_\xi$. Restricting to the combination of X^I's which are made out of Z_α and $\bar{Z}^\alpha$ then means that we project onto states made out of linear combination of $(X^I)^{\pm}$ fields for which the product of their $U(1)_\lambda \times U(1)_\xi$ is positive. In this way half of the degrees of freedom of X^I's are projected out. We perform a similar decomposition for the complex valued fermionic fields $\Psi^{\pm}$ which are in $\mathbf{8_s}$ of SO(8) and decompose them into $\mathbf{4}_{+1} + \bar{\mathbf{4}}_{-1}$ of $SU(4) \times U(1)_\xi$ fermions and work with the states made out of linear combinations of Ψ's the product of their $U(1)_\lambda \times U(1)_\xi$ charges is $+1$.

The supersymmetry variation parameters ϵ do not carry $\pm$ indices (they are neutral under $U(1)_\lambda$) and are in $\mathbf{8}_c$ of SO(8), as well as being a $3d$ anti-Majorana fermion. The $\mathbf{8}_c$ decomposes to $\mathbf{6}_0 + \mathbf{1}_{-2} + \mathbf{1}_{+2}$ of $SU(4) \times U(1)_\xi$. If together with working with the configurations (states) which are made out of Z_α and its fermionic counterpart, we *restrict* ourselves to the supersymmetry transformations generated by ϵ which are in $\mathbf{6}_0$, χ-terms

[8] To render the action invariant, there is one other option: To restrict the theory to specific (BPS) configurations over which $\partial_\mu J^\mu$ vanishes. These specific configurations should, however, form a *closed sector* in the Hilbert space. We will briefly explore this possibility in section 6.

JHEP12(2008)111

vanish. To see this let us consider χ^+_{AB}. Vanishing of $\bar{\epsilon}_2\chi^+_{AB}\epsilon_1$ may be seen recalling the form of $\chi^\pm_{AB}$ (4.28) and noting that the V_{JK} part is in $(\mathbf{6}\times\mathbf{6})_{A.S.} = \mathbf{15}$ while the $X^J X^K$ piece is in $(\mathbf{4}_{+1}\times\mathbf{4}_{+1})_{A.S.} = \mathbf{6}_{+2}$. Since $\mathbf{15}\times\mathbf{6}$ does not give a singlet of SU(4), $\bar{\epsilon}_2\chi^+\epsilon_1$ vanishes. Similarly one can argue that χ^- vanishes. In this way out of 16 independent fermionic transformations only 12 of them close onto the supersymmetry algebra.

To summarize, restricting the fields to Z_α and their fermionic counterpart the supersymmetry transformations which are generated by ϵ's in $\mathbf{6}_0$ of $\mathrm{SU}(4)\times\mathrm{U}(1)_\xi$ close and our gauge invariant action will describe a theory which has $3d$, $\mathcal{N}=6$ supersymmetry.

Although for a generic configuration we are dealing with an $\mathcal{N}=6$ theory, there are still large class of states (configurations) which exhibit more fermionic symmetries than expected from the $\mathcal{N}=6$ theory. Let us consider states of the form $\mathcal{O}^{I_1\cdots I_l} = Tr(X^{I_1}X^{I_2}\cdots X^{I_l})$ where the trace is over the $2N\times 2N$ matrices.[9] It is a straightforward computation to show that under two successive supersymmetry transformations $[\delta_1,\delta_2]\mathcal{O}_{I_1\cdots I_l} = v^\mu\partial_\mu\mathcal{O}_{I_1\cdots I_l}$. One can repeat the same computation with operators in which some of the X^I's are replaced with SO(8) fermions Ψ. For these operators, too, two successive supersymmetry transformations close onto the derivative of the operator. For the operators which involve covariant derivative of X^I or Ψ, e.g. $Tr(X^I D_\mu X^J)$, the supersymmetry does not close onto translations; for these operators there remain some terms stemming from the $\chi^\pm$ terms in (4.34). We note that the set of $\mathcal{O}_{I_1\cdots I_l}$ type operators include the chiral primaries. Therefore, although in general our theory enjoys $\mathcal{N}=6$ supersymmetry, there are large classes of gauge invariant BPS states which can preserve more fermionic symmetries than the ones expected from an $\mathcal{N}=6$ theory. In section 6 we will discuss examples of such BPS states.

We point out that if we rewrite the action implementing the restriction of the fields to $\mathbf{4}_{+1}$ and $\bar{\mathbf{4}}_{-1}$ our theory reduces to the representation of the ABJM model in terms of (non-totally antisymmetric) three-algebras [20]. The structure constants of their model is hence equal to our f^{ABCD}. In this construction the $T^A_\pm$ are not appearing explicitly and one only deals with $N\times N$ matrices.

Before closing this section we stress that as discussed in section 3 the $N=2$ case is special in the sense that the f^{abcd} coefficients (3.15) vanish. As shown in the appendix B, our $u(2)$-based extended three-algebra is a double copy of the $so(4)$-based Bagger-Lambert three-algebra. One can use this observation to project out half of the excessive degrees of freedom of the $u(2)$ theory. Projecting onto the $\Phi^+_a = \Phi^-_a$, $\Phi^+_0 = -\Phi^-_0$ sector (where Φ is X^I or Ψ) and $a=1,2,3$, our theory reduces to the Bagger-Lambert theory. This projection explicitly keeps the SO(8) invariance as well as supersymmetry. (After this projection one may explicitly check that for this case there is a gauge, the one worked out in [3], in which the action becomes invariant under 16 supersymmetry transformations.) We emphasize

[9] Recalling that $(\sigma^+)^2 = (\sigma^-)^2 = 0$ and that $\sigma^\pm$ are traceless for odd l $\mathcal{O}^{I_1\cdots I_l}$ vanishes and for even l

$$\mathcal{O}^{I_1\cdots I_l} = Tr_N\left((X_+^{I_1}X_-^{I_2}X_+^{I_3}X_-^{I_4}\cdots X_-^{I_l}) + (X_-^{I_1}X_+^{I_2}X_-^{I_3}X_+^{I_4}\cdots X_+^{I_l})\right),$$

where Tr_N is over $N\times N$ matrices. Gauge invariant operators which are constructed out of trace over $2N\times 2N$ matrices are neutral under the $\mathrm{U}(1)_\lambda$. Moreover, $\mathcal{O}_{I_1\cdots I_l}$ type operators are also parity invariant.

that this is a different projection than the SU(4) × U(1) invariant one used to earlier. In this sense our analysis shows how the Bagger-Lambert and ABJM theories for $N = 2$ are different.

5. The $su(N) \times su(N)$ Chern-Simons representation

As argued the closure of the extended three-algebra, with the working assumption that t^A are generators of a (semi)-simple Lie algebra, fixes the Lie-algebra to be $u(N)$ in its $N \times N$ representation. Here we rewrite the theory using the explicit representation of f^{ABCD} in terms of $su(N)$ f and d tensors and remove the four-brackets. Let us start with the gauge fields $A_{\mu AB}$ and $\tilde{A}_{\mu AB}$. Using (3.15), (4.7) can be written as

$$\begin{aligned}
&\tilde{A}_{\mu cd} = f_{cde} A_{\mu e} + i d_{cde} B_{\mu e}, \\
&\tilde{A}_{\mu a0} = \tilde{A}_{\mu 0a} = \frac{2i}{\sqrt{2N}}\, B_{\mu a}, \\
&\tilde{A}_{\mu 00} = 0\,, \qquad\qquad \sum_a \tilde{A}_{\mu aa} = \sum_A \tilde{A}_{\mu AA} = 0
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
A_{\mu e} &\equiv -\frac{i}{12}\left(d_{abe} A_{\mu ab} + \frac{2}{\sqrt{2N}} (A_{\mu 0e} + A_{\mu e0}) \right)\,, \\
B_{\mu e} &\equiv -\frac{1}{12} f_{abe} A_{\mu ab},
\end{aligned} \tag{5.2}$$

are two *real* $su(N)$ *valued* gauge fields. The reality of $A_{\mu a}$ and $B_{\mu a}$ gauge fields is a result of (4.6).

The covariant derivative of the matter fields Φ in terms of these $su(N)$ gauge fields take the form

$$\begin{aligned}
(D_\mu \Phi)^+_d &= \partial_\mu \Phi^+_d - \left(f_{cde} A_{\mu e} + i d_{cde} B_{\mu e} \right) \Phi^+_c - \frac{2i}{\sqrt{2N}} B_{\mu d} \Phi^+_0\,, \\
(D_\mu \Phi)^+_0 &= \partial_\mu \Phi^+_0 - \frac{2i}{\sqrt{2N}} B_{\mu c} \Phi^+_c\ .
\end{aligned} \tag{5.3}$$

Note that $(D_\mu\Phi)^-_A = \left((D_\mu \Phi)^+_A \right)^*$.

Recalling the behavior of the gauge field under parity (4.13), we learn that under parity $A_{\mu a}$ behaves as a vector while $B_{\mu a}$ transforms as a pseudovector. Rewriting the twisted Chern-Simons part of the action in terms of A and B gauge fields we find

$$\mathcal{L}_{Chern-Simons} = \frac{1}{2} \epsilon_{\mu\nu\alpha} \left[-12 B_{\mu a} \partial_\nu A_{\alpha a} + 2 f_{abc} (B_{\mu a} B_{\nu b} B_{\alpha c} + 3 B_{\mu a} A_{\nu b} A_{\alpha c}) \right]\ . \tag{5.4}$$

With a vector $A_{\mu a}$ and pseudovector $B_{\mu a}$ it is clear that the above action is parity invariant.

Upon the field redefinition

$$\begin{aligned}
R_{\mu a} &= A_{\mu a} - B_{\mu a}, \\
L_{\mu a} &= A_{\mu a} + B_{\mu a}\,,
\end{aligned} \tag{5.5}$$

$$\mathcal{L}_{Chern-Simons} = \mathcal{L}_{Chern-Simons\ R} - \mathcal{L}_{Chern-Simons\ L}\,, \tag{5.6}$$

JHEP12(2008)111

where

$$\mathcal{L}_{Chern-Simons\ R} = \frac{3}{2}\ \epsilon_{\mu\nu\alpha} \left[R_{\mu a}\partial_\nu R_{\alpha a} - \frac{1}{3} f_{abc} R_{\mu a} R_{\nu b} R_{\alpha c} \right], \tag{5.7}$$

and similarly for $\mathcal{L}_{Chern-Simons\ L}$. Therefore, the Chern-Simons part of the action (4.3) is nothing but the standard $su(N) \times su(N)$ Chern-Simons action. The level of the Chern-Simons of the two $su(N)$ Chern-Simons factors are equal but with the opposite sign and the parity exchanges the two $su(N)$ factors. Under the $\mathrm{U}(1)_\lambda$, $A_{\mu AB}$ remains invariant (cf. (4.21)) and as a result the $su(N)$ gauge fields R_α and L_α also remain invariant.

In usual conventions for the Chern-Simons theories, we have obtained a Chern-Simons theory at level 12π. There is the possibility of getting the Chern-Simons theory in an arbitrary level $k \in \mathbb{Z}$. In order this we may keep the form of the action we start from (4.3), and similar to [26], replace the structure constants f^{ABCD} by $f^{ABCD}/12\pi k$ (this scaling does not change the fundamental identity and closure conditions). This may be achieved by changing the normalization of the $u(N)$ generators t^A to $t^A/\sqrt{12\pi k}$.

Starting from (4.9c), after appropriate decomposition of the gauge transformation parameter $\tilde{\Lambda}_{AB}$ and using the $su(N)$ identities listed in the appendix A, one can work out the behavior of the R_μ and L_μ gauge fields under gauge transformation

$$\begin{aligned} \delta_{\text{gauge}} R_{\mu a} &= \partial_\mu \rho_a - f_{abc} R_{\mu b} \rho_c, \\ \delta_{\text{gauge}} L_{\mu a} &= \partial_\mu \lambda_a - f_{abc} L_{\mu b} \lambda_c, \end{aligned} \tag{5.8}$$

which as expected are two $su(N)$ gauge transformations.

Now let us study behavior of the matter fields under the above $su(N) \times su(N)$ factors. From (4.9b) and after straightforward, but lengthy algebra using $su(N)$ identities listed in the appendix A, we find that

$$\begin{aligned} \delta_{\text{gauge}} \Phi^+ &= i[\chi^1, \Phi^+] + i\{\chi^2, \Phi^+\} \\ \delta_{\text{gauge}} \Phi^- &= i[\chi^1, \Phi^-] - i\{\chi^2, \Phi^-\}\,, \end{aligned} \tag{5.9}$$

where $\Phi^\pm$ includes both the $su(N)$ and $u(1)$ components, respectively $\Phi^\pm_a$ and $\Phi^\pm_0$, of the fields and

$$\chi^1_a = \frac{1}{2}(\rho_a - \lambda_a)\,, \qquad \chi^2_a = \frac{1}{2}(\rho_a + \lambda_a)\ . \tag{5.10}$$

Note that χ^i, like λ and ρ, are $su(N)$ (and not $u(N)$) valued. From (5.9) one can read the form of the finite gauge transformations of $\Phi^\pm$:

$$\Phi^+ \longrightarrow\ \tilde{\Phi}^+ = e^{i\lambda}\ \Phi^+\ e^{-i\rho}\,, \qquad \Phi^- \longrightarrow\ \tilde{\Phi}^- = e^{i\rho}\ \Phi^-\ e^{-i\lambda}\ . \tag{5.11}$$

That is, $\Phi^\pm$ are in the bi-fundamental representation of $su(N) \times su(N)$. As discussed under the (global) $\mathrm{U}(1)_\lambda$ $\Phi^\pm$ carry charge ± 1.

For completeness we also present the explicit form of the fermionic (supersymmetry) transformations in terms of the Chern-Simons fields. The scalars and fermions have basically the same form as given in (4.22a,b) and for the gauge fields (4.22c) becomes

$$\begin{aligned} \delta_{\text{susy}} A_\mu &= \frac{1}{12}\bar{\epsilon}\gamma_\mu\Gamma^I\big(\{(X^I)^+, \Psi^-\} - \{(X^I)^-, \Psi^+\}\big) \\ \delta_{\text{susy}} B_\mu &= -\frac{i}{12}\bar{\epsilon}\gamma_\mu\Gamma^I\big([(X^I)^+, \Psi^-] + [(X^I)^-, \Psi^+]\big)\ . \end{aligned} \tag{5.12}$$

JHEP12(2008)111

It should be noted that the above will become supersymmetry transformations once the projection to SU(4) $\times$ U(1) sector is performed.

So far we have presented our model in terms of an $su(N)\times su(N)$ Chern-Simons theory with explicit SO(8) $\times$ U(1)$_\lambda$ symmetry. As argued in previous section out of the 16 independent fermionic variations introduced in (4.22) only 12 of them can lead to symmetries of the action. The supersymmetry closes only on a SU(4) $\times$ U(1)$_\xi$ $\times$ U(1)$_\lambda$ invariant sector of the physical Fock space of the theory. Once the theory is rewritten in terms of the fields over which the supersymmetry closes our theory becomes the $3d$, $\mathcal{N}=6$ $su(N)\times su(N)$ Chern-Simons theory. Our theory is hence closely related to the ABJM model.[10]

The model ABJM proposed to describe the low energy dynamics of N M2-branes (on $\mathbf{C}^4/\mathbb{Z}_k$ orbifold) is, however, a $u(N)\times u(N)$ theory (rather than $su(N)\times su(N)$). This model is related to our model upon gauging two extra global U(1)'s. One of them is the U(1)$_\lambda$ and the other is the "center of mass" U(1), U(1)$_{cm}$. Recalling that $\Phi^\pm$ fileds are in the bi-fundamental of $su(N)\times su(N)$ (5.11), one may simply gauge the $U(1)_{cm}$ symmetry without the need to add any additional interactions for Φ's, once we identify the U(1)$_{cm}$ with the diagonal part of the $u(1)$'s in $u(N)\times u(N)$. Gauging U(1)$_{cm}$, then only amounts to adding the corresponding U(1) Chern-Simons term. As discussed in section 4.3.2 the U(1)$_\lambda$ charge changes sign under parity while the U(1)$_{cm}$ charge remains invariant. This is compatible with identifying U(1)$_{cm}$ with the diagonal U(1) and U(1)$_\lambda$ is the anti-symmetric combinations of the two U(1)'s in U(N) $\times$ U(N).[11] In the theory in which U(1)$_{cm}$ is gauged, even after fixing the gauge, we remain with a $\mathbb{Z}_k$ part of the U(1) and hence the Z_α are defined up to $\mathbb{Z}_k$ rotations. Therefore, this theory describes M2-branes on $\mathbf{C}^4/\mathbb{Z}_k$ orbifold. As discussed in [19], just gauging the two extra U(1)'s does not bring our $su(N)\times su(N)$ theory to the ABJM model and one should consider two points: U(N) $\simeq$ (SU(N)$\times$U(1))$/Z_N$ and that in the SU(N)$\times$U(1) theory, despite the fact that in general the Chern-Simons levels for the U(1) and SU(N) parts could be different, in the U(N) theory they are taken to be equal.

After relating our theory to the ABJM model, their arguments for the physical states also apply to ours. Physical states of our theory can be those which are invariant under U(1)$_\lambda$. In the language of our three-algebra representation, these states could be constructed by taking trace over $2N\times 2N$ matrices, like the $\mathcal{O}^{I_1 I_2\cdots I_l}$ operators of last section.[12] As discussed in [19], there are also states which carry k units of the U(1)$_\lambda$ charge,

[10]We should, however, note that as discussed earlier the Z_α and $\bar{Z}_\alpha$ are not related by worldvolume parity; they are related by a product of parity and U(1)$_\xi$ charge conjugation. In this sense the ABJM theory, even for $N=2$ is different than the Bagger-Lambert theory.

[11]As argued in [19] the $3d$ Chern-Simons U(1) gauge theory has the peculiar feature that its equation of motion is ${}^*F=J$ (J is the U(1) currents) and hence we have a global symmetry generated by the conserved current $J={}^*F$. The U(1)$_b$ symmetry in the ABJM model, which is a part of the R-symmetry of the M2-brane theory, is the global U(1) generated by the diagonal U(1) part of the U(N) $\times$ U(N) gauge symmetry (the U(1)$_\xi$ in our notation) through $J_d={}^*F_d$. We thank Ofer Aharony for clarifying comment on this point.

[12]It is instructive to note that the bi-fundamental nature of the ABJM fields Z_α, dictating that the gauge invariant combinations should involve $Z_\alpha\bar{Z}_\beta$ or $\bar{Z}_\alpha Z_\beta$ which fall into adjoint representations of either of the U(N) factors, is naturally encoded in our $2N\times 2N$ matrices. This is because of (3.6) which implies that $X^I X^J=(X^I)^+(X^J)^-+(X^J)^-(X^I)^+$.

JHEP12(2008)111

those which have particular Wilson lines attached.

We note that after gauging the two U(1)'s the theory cannot be expressed in terms of the (extended) three-algebra anymore.

6. Relation to the theory of M2-branes

The $3d$, $\mathcal{N} = 8$ (or its $\mathcal{N} = 6$ version) SCFT should arise as the low energy effective field theory limit of coincident multi M2-branes on flat space (or its orbifold). Here we argue that the action (4.3) for our $u(N)$-based extended three-algebra and after restricting ("projecting") to SU(4) invariant sector of the Hilbert space, describes theory of N M2-branes. In addition we bring arguments clarifying the need for the projection.

6.1 Pair-wise M2-brane picture

It is well known and understood that when N D-branes of string theories sit on top of each other we see the structure of a $u(N)$ gauge theory [25]. For the special case of D3-branes this theory (in the low energy limit) is the $u(N)$ $4d$ SCFT. The enhancement of the gauge symmetry to $u(N)$ in the D-brane case is facilitated by the (perturbative) description of D-branes in terms of open strings ending on or stretched between D-branes. In the coincident limit the lowest modes of these open strings become massless and hence cause the gauge symmetry enhancement (inverse of Higgs mechanism). The above picture for D-branes and open strings stretched between them is valid for any pair of D-branes in a system of N D-branes [25].

The above "pair-wise" picture does not readily generalize to the M2-branes, as here we do not have the open strings picture. Nonetheless, we have open membranes stretched between two M2-branes. To see how these open membranes come about, let us start with two parallel D-branes in $10d$ IIA string theory. As shown in figure 1A there are (virtual) open string anti-string pairs stretched between the D2-branes. These open strings are oriented and the difference between the open string and anti-open string is the orientation; they are related by the worldsheet parity. When uplifted to M-theory the D2-branes become M2-branes while the stretched open strings become open membranes and anti-open membranes (see figure 1B).

Had we directly started in the $11d$ M-theory, as membrane worldvolume have two spatial directions, unlike the string case and as depicted in figure 2, there are two distinct options for open membrane anti-open membrane pair. These two pairs are related by the worldvolume parity. On the other hand, from the M2-brane viewpoint not all the four possibilities in the figure 2 are independent, explicitly, A and D open membranes and B and C open membranes cannot be distinguished by their M2-brane charge.

In the same spirit as D-branes, for the case of N M2-branes, we expect that we should be dealing with $2N \times 2N$ matrices. In our realization the 2×2 $\sigma^{\pm}$ part of the $T^A_{\pm}$ generators basically account for this "doubling" of the degrees of freedom corresponding to the open membrane pairs (compared to the open string case). However, as discussed not all the degrees of freedom of these stretched membranes are physically independent and moreover, not all of them can appear in the supersymmetric Fock space of the M2-brane theory; we

JHEP12(2008)111

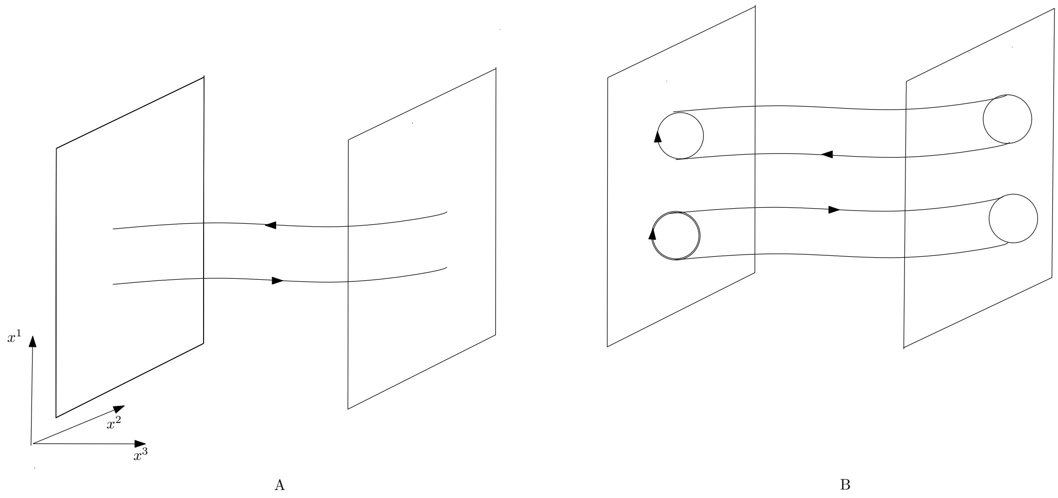

Figure 1: The figure on the left shows the open string anti-open string pair stretched between two parallel $D2$-branes along 012 directions while separated in x^3. The figure on the right shows the same system after uplifting to M-theory, where the open string pair now appear as open membrane anti-open membrane pair. Note that the open membranes wrap the 11^{th} circle in the same orientation.

need to mod out half of them. Restricting to the sector over which the supersymmetry transformations close (onto the $3d$, $\mathcal{N}=6$) these extra degrees of freedom are removed. This sector is identified with part of the Fock space, the physical Fock space, which is made out of functions of combinations of X's and Ψ's which the $\mathrm{U}(1)_\lambda$ and $\mathrm{U}(1)_\xi$ have the same sign. In addition, this picture also sheds light on the $su(N)\times su(N)$ structure.

Starting from this M2-brane picture, compactifying down to $10d$ IIA theory, however, only one of the two open membrane pairs survive the supersymmetry requirement. Supersymmetry demands that open membrane pairs should have the same orientation on the 11^{th} circle (in Fig 2, i.e. A and B or C and D pair). Therefore, at the IIA and D2-brane level we only see a single $su(N)$ factor.[13]

6.2 Analysis of BPS states

In the previous subsection, based on the stretched open membrane picture, we argued that we expect an $su(N)\times su(N)$ Chern-Simons theory (of course plus the gauging of the two extra $u(1)$'s) to describe N M2-branes (on an orbifold). To substantiate this result we analyze the BPS states of our theory.

Recall that not all the generic configurations of our X^I and Ψ fields close the supersymmetry "algebra" resulting from the fermionic transformations (4.22). As discussed all the (bosonic) configurations which are formed out of Z_α fall into representations of $\mathcal{N}=6$ algebra. However, there could be some states preserving more supersymmetry than expected from the $\mathcal{N}=6$ theory. In order not to lose the extra supersymmetry of these states, we perform the BPS analysis as follows. First we find solutions to $\delta_{\rm susy}\Upsilon=0$, with

[13]We should stress that, since we do not have the spectrum of open membranes, unlike the case of strings, our open membrane picture should be only taken as a helpful and suggestive pictorial way of presenting the $su(N)\times su(N)$ structure.

JHEP12(2008)111

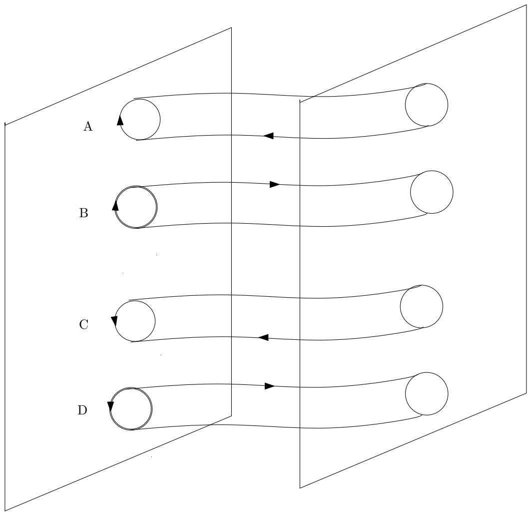

Figure 2: There are two options for open membrane anti-open membrane pairs stretched between two M2-branes. If we assume the M2-branes to be along 012 directions, these open membranes are along say 034 (x^3 is the direction the M2-branes are separated and x^4 is along the circular part of the open membranes). The open membrane pair A and B are mapped to C and D under worldvolume parity. Note that the circular direction on the open membranes is just for illustrative purposes and in terms of our matrices this part is associated with the $\sigma^{\pm}$ parts. In terms of what we have in the figure, i.e. A and D are associated with T_+^A and B and C membranes with T_-^A. Note that as far as the M2-brane charge is concerned the A and D and, B and C open membranes are indistinguishable and hence we need to mod out the "excess of degrees of freedom" we have introduced in our setting.

Υ being either of X^I, Ψ, or $A_{\mu AB}$ fields and ignoring the fact that not all the configurations which satisfy $\delta\Upsilon = 0$ are necessarily falling into the representations of $\mathcal{N} = 8$ or $\mathcal{N} = 6$ superPoincaré algebra. As the second step we check whether these particular (BPS) configurations/states indeed satisfy the closure of supersymmetry algebra. In order this we check if $[\delta_1, \delta_2]$, with δ given in (4.22), on the specific configuration in question is equal to $v^\mu \partial_\mu$ on that configuration.

6.2.1 Half-BPS states

As the candidate for N M2-branes on the $11d$ flat space (or its orbifold) the moduli space of 1/2 BPS configurations of our model must be R^{8N}/S_N (or $(\mathbf{C}^4/\mathbf{Z}_k)^N/S_N$). The half BPS sector of our model is the one for which the right-hand-side of all supersymmetry variations (4.22) vanishes for any arbitrary fermionic transformation parameter ϵ. Variations of the bosonic fields identically vanish for a pure bosonic configuration. Variation of fermions vanish for arbitrary ϵ only when the two terms in $\delta\Psi$ vanish independently, i.e.

$$D_\mu X^I = 0\,, \tag{6.1a}$$

$$[X^I, X^J, X^K, T] = 0. \tag{6.1b}$$

JHEP12(2008)111

When (6.1a) holds and the fermionic fields are turned off, the equations of motion for the two $su(N)$ gauge fields imply that both the gauge fields have flat connection and hence they can be set to zero in appropriate gauge. In this gauge, (6.1a) implies $\partial_\mu X^I = 0$. (6.1b) is satisfied if and only if

$$[(X^I)^+, (X^J)^-] = 0\,, \qquad [(X^I)^+, (X^J)^+] = 0\,, \tag{6.2}$$

where $(X^I)^\pm$ are the $N \times N$ matrices and may be defined through taking trace over 2×2 parts of the $2N \times 2N$ matrices, explicitly: $(X^I)^\pm = Tr_{2\times 2}\left(X^I \cdot (\mathbb{1}_N \otimes \sigma^\mp)\right)$. (6.2) is satisfied for any diagonal $N \times N$ matrices (on the elements on the diagonal complex valued). To find the moduli space of physical solutions, however, we still need to restrict ourselves to the $\mathcal{N} = 6$ supersymmetric sector. This is done by restricting to diagonal Z_α matrices. This removes half of the solutions, rendering the solutions to $8N$ real parameters. The analysis then becomes identical to that of ABJM [19] with a minor difference on the number of conserved supercharges: Recalling (4.29) and (6.1), it is readily seen that $[\delta_1, \delta_2]$ over these configurations vanish. Moreover, for these configurations and also the other states which fall into the same $\mathcal{N} = 8$ supermultiplet the variation of the action (4.24) vanishes. Therefore, these configurations form a sector which is invariant under all the 16 "supersymmetry" variations are 1/2 BPS in the sense of $\mathcal{N} = 8$.

6.2.2 1/4-BPS, Basu-Harvey configuration

There are much further options for less BPS cases. Here we consider the 1/4 BPS state which corresponds to M2-brane along 056 ending on an M5-brane along 012345, the Basu-Harvey configuration [27].[14] Turning off the fermions, the BPS configurations are obtained as solutions to $\delta\Psi = 0$. Let us turn on X^I, $I = 1, 2, 3, 4$, while setting X^I, $I = 5, 6, 7, 8$, to zero and denote non-zero X's as X^i, $i = 1, 2, 3, 4$. The BPS equation takes the form

$$\left(\gamma^\mu \Gamma^i D_\mu X^i + \frac{1}{6}\Gamma^{ijk}[X^i, X^j, X^k, T]\right)\epsilon = 0\ . \tag{6.3}$$

The above is basically the Basu-Harvey equation [27]. Here we just review its solutions. Consider the configurations for which the gauge fields are vanishing and also take X^i to only depend on one of worldvolume coordinates, say x_2. The x dependence of the two terms in (6.3) can be factored out if and only if[15]

$$X^i = \frac{1}{\sqrt{2 \cdot |x_2 - x_2^0|}} J^i, \tag{6.4}$$

where x_2^0 is an integration constant and J^i are some (x-independent) matrices which should satisfy

$$\left(\gamma^2 \Gamma^i J^i - \frac{s}{6}[J^i, J^j, J^k, T]\Gamma^{ijk}\right)\epsilon = 0\,, \tag{6.5}$$

[14] For the analysis of finding M5-M2 solutions in the ABJM model see [28]. Analysis of some other BPS or time-dependent non-BPS configurations of the BLG or ABJM models may be found in [29].

[15] Note that in our conventions the scalar fields X^I have mass dimension 1/2, while fermions Ψ and gauge field $A_{\mu AB}$ have mass dimension 1.

JHEP12(2008)111

where $s = \frac{x_2 - x_2^0}{|x_2 - x_2^0|}$ is taking ± 1 values. The Γ^i are four of the SO(8) Majorana-Weyl Dirac matrices and hence can be viewed as SO(4) $\in$ SO(8) Dirac matrices and therefore

$$\Gamma^{ijk} = \epsilon^{ijkl}\Gamma^5\Gamma^l,$$

where Γ^5 is the SO(4) chirality matrix. ϵ is a two component $3d$ fermion, while also in $\mathbf{8}_c$ of SO(8) R-symmetry. As such,

$$\begin{aligned} \gamma^2\epsilon &= s_1\epsilon\,, \\ \Gamma^5\epsilon &= s_2\epsilon\,, \end{aligned} \tag{6.6}$$

where s_1 and s_2 can (independently) be $+1$ or -1. Inserting the above into (6.5) and after some simple algebra we arrive at

$$[J^i, J^j, J^k, T] = s s_1 s_2 \epsilon^{ijkl} J^l\ . \tag{6.7}$$

The above has a solution in terms of $2N \times 2N$ representation of SO(4), if we take J^i to be proportional to $2N \times 2N$ SO(4) Dirac matrices and T to be proportional to $2N \times 2N$ "Γ^5". (For a detailed discussion on constructing solutions of (6.7) see [21].) Moreover, for $s = +1$ (i.e. for $x_2 > x_2^0$) we should take $s_1 s_2 = -1$ and for $s = -1$ $s_1 s_2 = +1$. Let us focus on the $s = +1$ for which there are two types of solutions, $s_1 = +1,\ s_2 = -1$, or $s_1 = -1,\ s_2 = +1$, each of which are invariant under transformations generated by four independent ϵ's and hence altogether our solution is invariant under eight fermionic transformations. (For the $s = -1$, $x_2 < x_2^0$ case, there are again eight ϵ's.)

We should now check if our configurations indeed satisfy the closure of the two successive supersymmetry transformations. To see this we note that, $v^\mu \partial_\mu X^I = -2i\bar{\epsilon}_2\gamma^2\epsilon_1\partial_2 X^I$. On the other hand for our solutions ϵ_i are eigenstates of γ^2 (cf. (6.6)) and therefore, $v^\mu|_{\mu=2} = 0$. For the same reason V_{JK} (4.30) is zero and hence $[\delta_1, \delta_2]X^I = v^\mu\partial_\mu X^I = 0$. As a result our configuration is a 1/4 BPS configuration and preserves 8 supercharges.

It is instructive to also present the solution in terms of our earlier notation and $(X^i)^\pm$ components: $(X^i)^\pm \propto (1 \pm T)\mathcal{J}^i$, where $T = \mathcal{J}^5$ and $\mathcal{J}^i$'s are $2N \times 2N$ SO(4) Dirac γ-matrices [21]. It is evident that $((X^i)^+)^\dagger = (X^i)^-$ and moreover for our solution $X^i_+ = X^i_-$. In terms of the ABJM complex Z_α fields our solution is $Z_\alpha = \bar{Z}_\alpha = X^i$. Note also that our solution is invariant under parity.

7. Discussion

In this work we have attempted generalizing the $3d$, $\mathcal{N} = 8$ BLG gauge theory by extending the notion of three-algebras. As we argued invariance of the BLG action under gauge symmetry requires a weaker condition than what is demanded by Bagger-Lambert (BL) three-algebras. In particular, in this work we focused on a notion of *extended fundamental identity*. Based on this notion we constructed an extended three-algebra, while giving a representation of the BL three-brackets in terms of an explicitly totally antisymmetric four-bracket and an explicit matrix representation for the algebra elements.

JHEP12(2008)111

We showed that the closure of our extended three-algebra, under the working assumption that t^A (3.2) are generators of a (semi-simple) Lie-algebra, fixes t^A to be generators of $u(N)$ in its $N \times N$ (fundamental) representation. We hence called this new three-algebra, the $u(N)$-based extended three-algebra. As we showed (see appendix C) the $N = 2$ case reproduces two copies of the BL $so(4)$-based three-algebra and in this sense our extended algebras are a generalization of BL three-algebras to $N > 2$ (in the M2-brane picture N is the number of M2-branes). It is interesting to explore whether one can relax this working assumption and study other kinds of extended three-algebras which may arise in this way and the BLG theory based on them.

We showed that the BLG theory for the $u(N)$-based extended three-algebra can be rewritten in terms of a $3d$ $su(N)\times su(N)$ Chern-Simons theory with SO(8) global symmetry and fields in the bi-fundamentals of the $su(N) \times su(N)$. Our theory, however, has twice more than the expected physical degrees of freedom. The bi-fundamental fields appear as a direct result of our choice of $2N \times 2N$ matrices (cf. footnote 12).

This theory, although invariant under the $3d$ parity, involves propagating scalar fields which are not parity invariant. To reduces the number of scalar degrees of freedom to the desired one, half of the existing ones, and also to close the fermionic variations onto a supersymmetry algebra, we projected the states onto the $\mathrm{SU}(4) \times \mathrm{U}(1) \in \mathrm{SO}(8)$ sector of the Hilbert space which is invariant under the parity times the $\mathrm{U}(1)_\xi$ charge conjugation. After this projection the theory becomes an $\mathcal{N} = 6$ $su(N) \times su(N)$ Chern-Simons theory. We discussed connection of our model with that of ABJM [19]. As discussed, for the special $N = 2$ case there is another way of projecting out half of the extra degrees of freedom in an SO(8) invariant manner and obtain the original Bagger-Lambert theory. It is interesting to see if there are other ways of projecting the extra degrees of freedom by the other discrete symmetries of our problem and obtain other $3d$ supersymmetric possibly SO(8) invariant Chern-Simons theories.

Although the $\mathcal{N} = 6$ Chern-Simons theory is very restrictive [30], there are other possibilities (than $su(N) \times su(N)$) for the gauge groups and matter content. Moreover, motivated by the ABJM model, recently many supersymmetric Chern-Simons theories with $\mathcal{N} \leq 5$ has been constructed (e.g. see [31] and references therein). As we showed, for the $\mathcal{N} = 6$ theories, the cases others than $su(N)\times su(N)$ theory does not have a representation in terms of our extended three-algebras. In this viewpoint the ABJM type theory is special. It is interesting to see whether within our extended algebras (presumably by relaxing the working assumption mentioned above) or within the "generalized Bagger-Lambert three-algebras" [20] these other cases also find a representation in terms of three-algebras. For a recent work in this direction see [32].

We gave a very suggestive picture for realization of $su(N) \times su(N)$ gauge group, our argument was a generalization or extension of the similar picture for D-branes. It is desirable to make our "pair-wise" picture more quantitative and see how the structure of the extended three-algebra may come out of this picture.

To provide further evidence one may also construct other BPS configurations and compare it against the result expected from a system of M2-branes. One may also compute the supersymmetric (Witten) indices for our $su(N) \times su(N)$ theory. The computation

JHEP12(2008)111

should closely follow that of the ABJM theory [35]. However, in our case we should also implement the "projection onto supersymmetric Hilbert space" in computation of the partition function or supersymmetric indices. Providing these further pieces of evidence in support of our proposed model is postponed to future works.

Acknowledgments

The author is indebted to Neil Lambert for his clarifying comments and critiques about the supersymmetry closure. I would like to thank Ofer Aharony, Mohammad Ali-Akbari and Joan Simon for fruitful comments or discussions.

A. Conventions and useful identities for $su(N)$ algebras

In our conventions, generators of the $u(N)$ algebra in its $N \times N$ (fundamental) representation are denoted by t^A, $A = 0, 1, \cdots, N^2 - 1$. Among t^A's, t^0 is the generator of $u(1)$ and t^a, $a = 1, 2, \cdots, N^2 - 1$, are generators of $su(N)$. In our normalization

$$Tr(t^A t^B) = \frac{1}{2}\delta^{AB}, \tag{A.1}$$

and therefore

$$t^0 = \frac{1}{\sqrt{2N}} \mathbb{1}_N. \tag{A.2}$$

The product of two generators:

$$\begin{aligned} t^a t^b &= \frac{i}{2} f^{abc}\ t^c + \frac{1}{2} d^{abc} t^c + \frac{1}{2N}\delta^{ab}\mathbb{1}_N\ , \\ t^0 t^a &= t^a t^0 = \frac{1}{\sqrt{2N}} t^a, \\ t^0 t^0 &= \frac{1}{2N}\mathbb{1}\ , \end{aligned} \tag{A.3}$$

where f^{abc} (which is totally anti-symmetric) is the structure constant of the $su(N)$ algebra and d^{abc} is the totally symmetric traceless tensor of $su(N)$. From the above it is seen that

$$\sum_A\ t^A t^A = \frac{N}{2}\mathbb{1}_N\ ,$$

and

$$[t^a, t^b] = i f^{abc} t^c, \qquad \{t^a, t^b\} = d^{abc} t^c + \frac{1}{N}\delta^{ab}\mathbb{1}\ .$$

Useful identities on the product of f's and d's: here we list some identities which have been used in computations performed in the main text. These identities are taken from [36].

- Product of two f's or d's:

$$\begin{aligned} f_{acd}\ f_{bcd} &= N\ \delta_{ab}\ , \\ f_{acd}\ d_{bcd} &= 0\ , \\ d_{acd}\ d_{bcd} &= \frac{N^2 - 4}{N}\ \delta_{ab}\ . \end{aligned} \tag{A.4}$$

JHEP12(2008)111

- The Jacobi identities

$$
\begin{aligned}
f_{ade}f_{bce} + f_{bde}f_{cae} + f_{cde}f_{abe} &= 0, \\
f_{ade}d_{bce} + f_{bde}d_{cae} + f_{cde}d_{abe} &= 0, \\
f_{abe}f_{cde} &= \frac{2}{N}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + (d_{ace}d_{bde} - d_{ade}d_{bce}) \ .
\end{aligned}
\tag{A.5}
$$

- Product of three f's or d's

$$
\begin{aligned}
f_{ade}f_{beg}f_{cgd} &= \frac{N}{2}f_{abc} \,, \\
d_{ade}f_{beg}f_{cgd} &= \frac{N}{2}d_{abc} \,, \\
d_{ade}d_{beg}f_{cgd} &= -\left(\frac{N^2-4}{2N}\right) f_{abc} \,, \\
d_{ade}d_{beg}d_{cgd} &= 3\left(\frac{N^4-4}{2N}\right) d_{abc} \ .
\end{aligned}
\tag{A.6}
$$

(For the last identity there is a typo in [36] which we have corrected.)

B. On the uniqueness of the $u(N)$-based extended three-algebras

Here we present line of arguments which show that with the working assumption that t^A are generators of semi-simple Lie-algebras, (3.11) can only hold for $u(N)$ in its $N \times N$ representation. Our argument is arranged in two steps:

I) For any finite dimensional matrix representation of simple Lie-algebra the generators are traceless (because trace of a commutator is zero). On the other hand one can always normalize the basis such that $Tr(t^A t^B) = \frac{1}{2}\delta_{AB}$, t^A being generators of any simple algebra. Trace of left-hand-side of (3.11) is not zero (it is just the structure constant of the algebra f^{ABC}). Therefore, t^A satisfying (3.11) cannot be generators of any simple *non-Abelian* Lie-algebra or direct products of thereof. Moreover, to satisfy (3.11) for a "semi-simple" Lie algebra generators it must contain *Abelian* factors.

II) One can show that in order (3.11) to hold, generically, the product of any two generators, and not only their commutators, should also be in the same algebra, i.e.

$$
\{t^A, t^B\} = F^{ABC}t^C, \tag{B.1}
$$

for some numeric coefficient expansions F^{ABC}. In the matrix representations, this latter only holds only for any generic $N \times N$ matrices and within our working assumption that is only $u(N)$ (or direct products of $u(N)$'s).

To see how (3.11) leads to (B.1), let us assume that we are working with $N \times N$ representation for t^A's and

$$
\{t^A, t^B\} = F^{ABC}t^C + G^{AB\alpha}X^\alpha, \tag{B.2}
$$

JHEP12(2008)111

where X^α are the set of all $N \times N$ matrices which cannot be expressed as linear combination of t^A's. In other words, X^α are "complementary" to t^A in covering the $N \times N$ matrices. Without loss of generality we may choose the X^α such that $Tr(t^A X^\alpha) = 0$, and let $Tr(X^\alpha X^\beta) = g^{\alpha\beta}$. Next, multiply both sides of (3.11) by X^α and take the trace. The right-hand-side vanishes while the left-hand-side does not; it vanishes only if $G^{AB\alpha} = 0$ (for any A, B, α) or $g^{\alpha\beta} = 0$ (for any α, β). The latter cannot happen because there is a simple counter-example: if the t^A are not generators of $u(N)$, then there are elements in the "complementary" set the trace of product of its generators are not zero. We then remain with $G^{AB\alpha} = 0$ choice which implies (B.1) and hence proving the statement.

C. $so(4)$-based Bagger-Lambert three-algebra as an extended three-algebra

As mentioned, in our construction the $u(N)$-based extended three-algebra is a metric three-algebra with a positive definite metric. This is readily seen from (3.7). (For the same reason we do not expect the Lorentzian $u(N)$ three-algebras to have a realization in terms of our extended three-algebras. Nonetheless, as discussed in [15], they do admit a representation in terms of matrices and four-brackets.) It is hence interesting to see if the $so(4)$-based Bagger-Lambert (BL) three-algebra can be obtained as a special case of our $u(N)$-based extended three-algebra.

The obvious candidate for realization of $so(4)$-based BL three-algebra is $u(2)$-based extended three-algebra. For this case the $T^A_\pm$ generators are

$$T^A_\pm = \frac{1}{2}\sigma^A \otimes \sigma_\pm \,, \qquad A = 0, 1, 2, 3. \tag{C.1}$$

where $\sigma^A = (\mathbb{1}_2, \sigma^a),\ a = 1, 2, 3$. The above are eight matrices and can be decomposed as

$$\begin{aligned} T^a_\pm &= \frac{1}{4}\gamma^a(1 \pm \gamma^5), \qquad a = 1, 2, 3\,, \\ T^0_\pm &= \pm\frac{1}{4i}\gamma^4(1 \mp \gamma^5), \end{aligned} \tag{C.2}$$

where $T = 1 \otimes \sigma^3 = \gamma^5$. $(T^a_+ + T^a_-)$ and $i(T^0_+ - T^0_-)$ combination of the $T^A_\pm$ matrices, are the $so(4)$ Dirac γ-matrices. In other words, if we restrict ourselves to the sector of the theory in which $\Phi^a_+ = \Phi^a_-$ and $\Phi^0_+ = -\Phi^0_-$, the $u(2)$-based extended three-algebra becomes the $so(4)$-based BL three-algebra written in another basis. There are, however, some comments:

1) The $su(2)$ algebra, among the $su(N)$ algebras, is special in the sense that its totally symmetric traceless three tensor d_{abc} identically vanishes (which is compatible with (A.4) and (A.6) identities). This brings about a great simplification in the structure constants f^{ABCD}.

2) As we can see among eight $T^A_\pm$ one can construct γ^μ and $\gamma^\mu\gamma^5$ ($\mu = 1, 2, 3, 4$) and one can restrict the elements of the algebra to have components along γ^μ or along

JHEP12(2008)111

$\gamma^\mu\gamma^5$. In this sense our $u(2)$-based extended three-algebra contains two copies of the $so(4)$-based Bagger-Lambert three-algebra. One can choose to work with one half, say the one spanned by γ^μ's, as they close onto a sub-three-algebra. In this subalgebra, the structure constants take the form $\epsilon^{\mu\nu\alpha\beta}$. For the same reason in the $u(2)$ case in this specific sector our "extended fundamental identity" becomes the standard fundamental identity.

D. Compatibility of supersymmetry and parity

Fermionic transformations (4.22) are compatible with parity if the following identities are satisfied

$$(\delta_{\rm susy} X^I)_{\rm parity} = \delta_{\rm susy}(X^I_{\rm parity})\,, \tag{D.1a}$$

$$(\delta_{\rm susy}\Psi)_{\rm parity} = \delta_{\rm susy}(\Psi_{\rm parity})\,, \tag{D.1b}$$

$$(\delta_{\rm susy}\tilde{A}_{\mu AB})_{\rm parity} = \delta_{\rm susy}(\tilde{A}_{\mu AB})^*\,, \tag{D.1c}$$

where in the last equality we have used (4.14), $X^I_{\rm parity}$, $\Psi_{\rm parity}$ are defined in (4.17) and note that under parity the supersymmetry parameter ϵ_p is transformed as[16]

$$\epsilon \longrightarrow \epsilon_p = -\gamma^2\epsilon\ . \tag{D.2}$$

With this choice $\bar\epsilon\ \Psi$ behaves as a scalar (rather than a pseudoscalar) and $\bar\epsilon\gamma^\mu\Psi$ behaves as a vector. Therefore, recalling (4.22a) and (4.17), (D.1a) becomes immediate.

To check (D.1b), we note that

$$\gamma^\mu(D_\mu X^I)_{\rm parity} = -\gamma^2\gamma^\mu D_\mu(X^I_{\rm parity})\gamma^2\,,$$

and hence the first term in $(\delta_{\rm susy}\Psi)$, goes to the first term in $\delta_{\rm susy}(\Psi_{\rm parity})$. Recalling (4.18) one finds that the second term in $\delta_{\rm susy}\Psi$ goes to $\delta_{\rm susy}(\Psi_{\rm parity})$. Putting these together we have:

$$(\delta_{\rm susy}\Psi)_{\rm parity} = \gamma^2\left(\gamma^\mu D_\mu X^I_{\rm parity}\Gamma^I\epsilon - \frac{1}{6}[X^I_{\rm parity}, X^J_{\rm parity}, X^K_{\rm parity}, T]\Gamma^{IJK}\epsilon\right).$$

which is nothing but (D.1b).

To verify (D.1c) we note that

$$\delta_{\rm susy}\tilde{A}_{\mu CD} = i\bar\epsilon\gamma_\mu\Gamma^I\Big((X^I)^+_A\ \Psi^-_B - (X^I)^-_B\ \Psi^+_A\Big)f_{ABCD}\,, \tag{D.3}$$

and hence

$$\begin{aligned}(\delta_{\rm susy}\tilde{A}_{\mu CD})_{\rm parity} &= i\bar\epsilon\gamma^2\gamma_\mu\gamma^2\Gamma^I\Big((X^I)^-_A\ \Psi^+_B - (X^I)^+_B\ \Psi^-_A\Big)f_{ABCD}\\ &= -i\bar\epsilon\gamma^2\gamma_\mu\gamma^2\Gamma^I\Big((X^I)^+_A\ \Psi^-_B - (X^I)^-_A\ \Psi^+_B\Big)f_{ABDC}\\ &= \delta_{\rm susy}(\tilde{A}_{\mu CD})_{\rm parity}\,,\end{aligned} \tag{D.4}$$

where in the second line of the above we have used (3.14) and in the third line (4.13). Note also that $\gamma^2\gamma_\mu\gamma^2 \equiv -\gamma^p_\mu$ where γ^p_μ is equal to γ^0, γ^1 for $\mu = 0, 1$ and to $-\gamma^2$ for $\mu = 2$.

[16] We would like to comment that the supersymmetry transformations of the Bagger-Lambert theory [3] are compatible with parity in the sense of (D.1) with the same choice for ϵ_p.

JHEP12(2008)111

References

[1] J.H. Schwarz, *Superconformal Chern-Simons theories*, *JHEP* **11** (2004) 078 [hep-th/0411077].

[2] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108].

[3] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple* $M2$*-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955]; *Comments on multiple* $M2$*-branes*, *JHEP* **02** (2008) 105 [arXiv:0712.3738].

[4] A. Gustavsson, *Algebraic structures on parallel* $M2$*-branes*, arXiv:0709.1260.

[5] A. Gustavsson, *Selfdual strings and loop space Nahm equations*, *JHEP* **04** (2008) 083 [arXiv:0802.3456].

[6] N. Lambert, *Lagrangians for multiple* $M2$*-braness*, talk presented at *Strings* 2008, August, Geneva, Switzwerland (2008), http://www.cern.ch/Strings2008.

[7] J.P. Gauntlett and J.B. Gutowski, *Constraining maximally supersymmetric membrane actions*, arXiv:0804.3078;
G. Papadopoulos, $M2$*-branes,* 3*-Lie algebras and Plücker relations*, *JHEP* **05** (2008) 054 [arXiv:0804.2662].

[8] J. Gomis, G. Milanesi and J.G. Russo, *Bagger-Lambert theory for general Lie algebras*, *JHEP* **06** (2008) 075 [arXiv:0805.1012].

[9] S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, $N = 8$ *superconformal gauge theories and* $M2$ *branes*, arXiv:0805.1087.

[10] P.-M. Ho, Y. Imamura and Y. Matsuo, $M2$ *to* $D2$ *revisited*, *JHEP* **07** (2008) 003 [arXiv:0805.1202].

[11] P. De Medeiros, J.M. Figueroa-O'Farrill and E. Mendez-Escobar, *Lorentzian Lie* 3*-algebras and their Bagger-Lambert moduli space*, *JHEP* **07** (2008) 111 [arXiv:0805.4363]; *Metric Lie* 3*-algebras in Bagger-Lambert theory*, *JHEP* **08** (2008) 045 [arXiv:0806.3242].

[12] S. Cecotti and A. Sen, *Coulomb branch of the lorentzian three algebra theory*, arXiv:0806.1990.

[13] S. Banerjee and A. Sen, *Interpreting the* $M2$*-brane action*, arXiv:0805.3930.

[14] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *Ghost-free superconformal action for multiple* $M2$*-branes*, *JHEP* **07** (2008) 117 [arXiv:0806.0054];
J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, *Supersymmetric Yang-Mills theory from lorentzian three-algebras*, *JHEP* **08** (2008) 094 [arXiv:0806.0738];
B. Ezhuthachan, S. Mukhi and C. Papageorgakis, $D2$ *to* $D2$, *JHEP* **07** (2008) 041 [arXiv:0806.1639].

[15] M. Ali-Akbari, M.M. Sheikh-Jabbari and J. Simon, *Relaxed three-algebras: their matrix representations and implications for multi* $M2$*-brane theory*, *JHEP* **12** (2008) 037 [arXiv:0807.1570].

[16] H. Verlinde, $D2$ *or* $M2$*? A note on membrane scattering*, arXiv:0807.2121.

[17] J.M. Maldacena, *The large-*N *limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

JHEP12(2008)111

[18] M. Van Raamsdonk, *Comments on the Bagger-Lambert theory and multiple $M2$-branes*, *JHEP* **05** (2008) 105 [`arXiv:0803.3803`].

[19] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, $N = 6$ *superconformal Chern-Simons-matter theories, $M2$-branes and their gravity duals*, *JHEP* **10** (2008) 091 [`arXiv:0806.1218`].

[20] J. Bagger and N. Lambert, *Three-algebras and $N = 6$ Chern-Simons gauge theories*, `arXiv:0807.0163`.

[21] M.M. Sheikh-Jabbari, *Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture*, *JHEP* **09** (2004) 017 [`hep-th/0406214`];
M.M. Sheikh-Jabbari and M. Torabian, *Classification of all* 1/2 *BPS solutions of the tiny graviton matrix theory*, *JHEP* **04** (2005) 001 [`hep-th/0501001`].

[22] L. Takhtajan, *On Foundation of the generalized Nambu mechanics (second version)*, *Commun. Math. Phys.* **160** (1994) 295 [`hep-th/9301111`].

[23] S. Cherkis and C. Sämann, *Multiple $M2$-branes and generalized* 3*-Lie algebras*, *Phys. Rev.* **D 78** (2008) 066019 [`arXiv:0807.0808`].

[24] P. de Medeiros, J. Figueroa-O'Farrill, E. Mendez-Escobar and P. Ritter, *On the Lie-algebraic origin of metric* 3*-algebras*, `arXiv:0809.1086`.

[25] E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys.* **B 460** (1996) 335 [`hep-th/9510135`].

[26] N. Lambert and D. Tong, *Membranes on an orbifold*, *Phys. Rev. Lett.* **101** (2008) 041602 [`arXiv:0804.1114`];
J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, *$M2$-branes on M-folds*, *JHEP* **05** (2008) 038 [`arXiv:0804.1256`].

[27] A. Basu and J.A. Harvey, *The $M2$-$M5$ brane system and a generalized Nahm's equation*, *Nucl. Phys.* **B 713** (2005) 136 [`hep-th/0412310`].

[28] S. Terashima, *On $M5$-branes in $N = 6$ membrane action*, *JHEP* **08** (2008) 080 [`arXiv:0807.0197`];
K. Hanaki and H. Lin, *$M2$-$M5$ systems in $N = 6$ Chern-Simons theory*, *JHEP* **09** (2008) 067 [`arXiv:0807.2074`].

[29] C. Krishnan and C. Maccaferri, *Membranes on calibrations*, *JHEP* **07** (2008) 005 [`arXiv:0805.3125`];
T. Fujimori, K. Iwasaki, Y. Kobayashi and S. Sasaki, *Time-dependent and non-BPS solutions in $N = 6$ superconformal Chern-Simons theory*, *JHEP* **12** (2008) 023 [`arXiv:0809.4778`].

[30] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, $N = 5, 6$ *superconformal Chern-Simons theories and $M2$-branes on orbifolds*, *JHEP* **09** (2008) 002 [`arXiv:0806.4977`];
M. Schnabl and Y. Tachikawa, *Classification of $N = 6$ superconformal theories of ABJM type*, `arXiv:0807.1102`.

[31] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, $N = 4$ *superconformal Chern-Simons theories with hyper and twisted hyper multiplets*, *JHEP* **07** (2008) 091 [`arXiv:0805.3662`];
O. Aharony, O. Bergman and D.L. Jafferis, *Fractional $M2$-branes*, *JHEP* **11** (2008) 043 [`arXiv:0807.4924`].

[32] M. Yamazaki, *Octonions, G_2 and generalized Lie* 3*-algebras*, *Phys. Lett.* **B 670** (2008) 215 [`arXiv:0809.1650`].

JHEP12(2008)111

[33] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, $N = 8$ *superconformal Chern-Simons theories*, *JHEP* **05** (2008) 025 [`arXiv:0803.3242`].

[34] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *Studies of the ABJM theory in a formulation with manifest* SU(4) *R-symmetry*, *JHEP* **09** (2008) 027 [`arXiv:0807.0880`].

[35] J. Bhattacharya and S. Minwalla, *Superconformal indices for* $\mathcal{N} = 6$ *Chern Simons theories*, `arXiv:0806.3251`.

[36] A.J. MacFarlane, A. Sudbery and P.H. Weisz, *On Gell-Mann's* λ*-matrices,* d *and* f *tensors, octets, and parametrizations of* SU(3), *Commun. Math. Phys.* **11** (1968) 77.

JHEP12(2008)111